Gacha Game Analysis and Design

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Gacha game is a special opaque selling approach, where the seller is selling gacha pulls to the buyer. Each gacha pull provides a certain probability for the buyer to win the gacha game reward. The gacha game has been enthusiastically embraced in numerous online video games and has a wide range of potential applications. In this work, we model the complex interaction between the seller and the buyer as a Stackelberg game, where the sequential decision of the buyer is modeled as a Markov Decision Process (MDP). We define the whale property in the context of gacha games. Then, we show that this is the necessary condition to achieve optimal revenue. Moreover, we provide the revenue-optimal gacha game design and show that it is equivalent to the single-item single-bidder Myerson auction. We further explore two popular multi-item gacha games, namely, the sequential multi-item gacha game and the banner-based multi-item gacha game. We also discuss the subsidies in the gacha game and demonstrate how subsidies may encourage the buyer to engage in grinding behavior. Finally, we provide a case study on blockchain systems as gacha games.

CCS Concepts: • **Applied computing → Marketing**; **E-commerce infrastructure**; • **General and reference → Design**.

Additional Key Words and Phrases: gacha game; e-commerce; probabilistic selling.

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1 INTRODUCTION

Gacha game [1] is a special opaque selling strategy, where the seller is selling gacha pulls to the buyer. Each gacha pull provides a certain probability for the buyer to win the gacha game, similar to a lottery ticket. Once the buyer wins the gacha game, he will receive the gacha game reward, e.g., a valuable item. Different from the straightforward lottery ticket, the probability to win the gacha game in each gacha pull can be varied.

One of the most popular applications of the gacha game is the video games. The gacha mechanic has been widely used in video games since the 2010s [42]. Most of these games are free-to-play (F2P) mobile games like "Genshin Impact", wherein the gacha mechanic is designed to incentivize players' in-game buying activities. Besides, the gacha game also has a wide range of applications in the realistic world, such as car plate lottery and probabilistic selling in e-commerce [30, 39, 46]. With the popularity of the gacha games, many interesting mechanisms have emerged, such as the varying-probability mechanism, reset-after-winning mechanism, and banner-based design, etc. With the widespread use of these interesting mechanisms, the underlying properties and the revenue guarantee remain mostly unclear.

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In this paper, we model the complex interaction between the seller and the buyer in the gacha game as a two-stage Stackelberg game, where the seller first designs the gacha game configuration including the price of each gacha pull and the winning probability in each round, and the buyer will then decide whether to buy the gacha pull from the seller. The buyer will engage in a consecutive interaction with the seller, and may buy a large number of gacha pulls sequentially. Such a sequential decision of the buyer can be modeled by a Markov Decision Process (MDP).

Next, we introduce a special kind of gacha game, gacha games with the whale property. For a whale property gacha game, the optimal policy for a rational buyer in that game is to either continue pulling the gacha until he wins or never pull the gacha. That is, the best response of a rational buyer is similar to "take-it-or-leave-it". On the contrary, the buyer in the non-whale property gacha game may pull the gacha game several times and then quit midway. One of our major findings is that the whale property is necessary for the seller's revenue maximization. We further show the equivalence of the gacha game and the single-item single-bidder Myerson auction [33] and figure out the optimal game configuration that can achieve the maximum seller's revenue.

We also investigate the gacha game where multiple items are sold. The multi-item gacha game includes multiple phases and each phase contains exactly one item. The buyer plays a gacha game in each phase. If the buyer wins the gacha game in the k-th phase, he will get the reward of that phase and then enter the next phase. One unique feature of the multi-item gacha game is that each phase could start at different states (or, winning probabilities), depending on how many times the buyer has pulled to get to the current phase. Such a history-dependent feature complicates the system design, and seems to degenerate the seller's revenue. A common practice is to introduce some "reset" mechanisms to let the buyer restart the pulling process at some point, e.g., reset the winning probability to the case as if the first pull after the buyer wins. Interestingly, with the two typical multi-item gacha game designs we analyzed, namely, the sequential gacha game and the banner-based gacha game, we will show that the idea of putting buyers back from the beginning has different impacts on different gacha games. The sequential gacha game captures the scenario where the buyer will pull the gacha game many times, which is common in many online video games. The reset-after-winning mechanism in the sequential gacha game will reset the buyer's state when the buyer wins the gacha game, as opposed to the succeed-after-winning mechanism. The popular F2P gacha game "Genshin Impact" adopts the reset-after-winning mechanism, while "Tower of Fantasy" adopts the succeed-after-winning mechanism. We show that the succeed-after-winning mechanism can achieve a higher seller's revenue compared to the reset-after-winning mechanism, while the reset-after-winning mechanism can achieve asymptotic optimality when there are infinite items. Besides, compared with the traditional bundle selling, which is also asymptotical optimal, the sequential gacha game can achieve a higher seller's revenue when considering buyer's budget constraint. The banner-based gacha game allows the buyer to opt-out the current phase, and will end a phase either the buyer wins the gacha game in the current phase or chooses to opt-out. This models the practical scenario where the items can only be acquired during a predetermined event time period, which is known as "banner". Each banner contains a specific item and works like a single-item gacha game. After a period of time, the next banner is released and replaces the current one. In the banner-based gacha game, the succeed-after-opt-out mechanism will carry the buyer's state from the current banner to the next banner, while the reset-after-opt-out mechanism will reset the buyer's state at the beginning of the next banner when the buyer chooses to opt-out. Then, we show that the following three selling mechanisms are equivalent in optimal revenue: (i) the reset-after-opt-out mechanism, (ii) the succeed-after-opt-out mechanism, and (iii) the separate selling with several independent single-item gacha games. Moreover, we find that the succeed-after-opt-out mechanism can achieve a higher seller's revenue when the buyers are budget-constrained.

This explains why the succeed-after-opt-out mechanism in the banner-based gacha game is popular in online video games.

Besides, we investigate the subsidies in the gacha game, where the seller will provide free gacha pulls to the buyer as subsidies. This is a common practice in many online video games [42]. Particularly, players in online video games can obtain some free gacha pulls by finishing some commissions, which motivates the players to play the game. Different from the previous research on subsidies in the market, where subsidy serves as a price discount [13], subsidies in the gacha game directly provide free gacha pulls to the buyers. With free gacha pulls, the buyers may get the reward of the gacha game for free without buying any gacha pulls from the seller, which may harm the seller's revenue. This property makes the subsidies in the gacha game more challenging. We show that subsidies in the varying-probability gacha game help the seller to further steer the buyer's behavior, and thus improve the seller's revenue. But we also show that the subsidies would harm the seller's revenue in the fixed-probability single-item gacha game and may lead to the buyer's grinding behavior in the banner-based multi-item gacha game.

Finally, we show that there is a wide range of applications of the gacha game, and the analysis and design of the gacha game can provide practical insights for these scenarios. Specifically, we provide a case study of the gacha game on the blockchain system. We model the blockchain system as a gacha game, where the buyers are the miners or validators, the seller is the system designer who wants to achieve the maximum security of the blockchain. The analysis of the gacha game can be applied to the blockchain system, and provide practical insights for the blockchain design.

Next, we provide a summary of our contributions and findings as follows:

(1) We propose the mathematical and systematical modeling framework for the gacha game, where the sequential decision of the buyer is modeled as a Markov Decision Process (MDP). Besides, we are the first to theoretically investigate the special characteristics and mechanisms of the gacha games including the whale property, the multi-item gacha game design and subsidies in the gacha game. The theoretical results of our proposed model are highly consistent with the empirical results of the previous sociological and psychological research.
(2) We introduce the definition of whale property in the gacha game and show the revenue optimality of the whale property. Besides, we show the equivalence of an arbitrary gacha game and the corresponding single-item single-bidder Myerson auction with explicit allocation rule and payment rule. We further provide the optimal gacha game configuration that can achieve the maximum seller's revenue.
(3) We explore two popular multi-item gacha game designs, namely, the sequential multi-item gacha game and the banner-based multi-item gacha game. We further discuss the reset-after-winning and succeed-after-winning mechanisms in the sequential gacha game, as well as the reset-after-opt-out and the succeed-after-opt-out mechanisms in the banner-based gacha game. Compared with the traditional multi-item selling methods, including bundle selling and separate selling, the multi-item gacha game design can achieve a higher seller's revenue when considering buyer's budget constraint.
(4) We study the subsidies in the gacha game, where the seller may provide free gacha pulls to the buyer. We show that when the buyer's valuation is too low, subsidies in the varying-probability gacha game can motivate the buyer to pull the gacha game and thus improve the seller's revenue. However, subsidies in the banner-based multi-item gacha game may lead to the buyer's grinding behavior, which may harm the seller's revenue.
(5) We discuss potential applications of the gacha game, and are the first to model the blockchain system as a gacha game, and provide practical insights for the blockchain design from the gacha game perspective.

This paper is organized as follows. Section 2 presents a review of the related work. In Section 3, we provide an overview of the modeling framework in gacha games. In Section 4, we investigate the revenue optimal single-item gacha game. In Section 5, we explore the multi-item gacha game. In Section 6, we discuss the subsidies and the buyer's grinding behavior in the gacha game. In Section 7, we provide a case study of the gacha game on blockchain systems. In Section 8, we conclude the paper with the final remark. Due to space limitations, the detailed calculation of the examples and the proofs of the theoretical results are provided in appendix.

2 LITERATURE REVIEW

Our work draws inspiration from and is related to several areas across social analysis, operation management, computer science, and economics.

Most of the previous research on the gacha game is from the sociological and psychological perspectives. These previous works empirically analyze the gacha game mainly from statistics, psychology, and society. Different from the previous work, we are the first to propose a mathematical and systematical modeling framework for the gacha game. In [24, 27, 29], the authors compare the gacha game to gambling. [35] proposes a psychoanalytical approach to analyze the gacha game. Besides, [44] claims that the gacha system is addictive and problematic. In our work, we introduce the whale property and explain why some buyers may continue pulling the gacha until they win the game. [9, 31] find that in the financing of gacha games, a large part of the game's revenue originates from a very small proportion of players who spend a large amount of money, essentially subsidizing the game for other players who may spend less money. This finding is consistent with the theoretical results of the optimality of the whale property gacha game. For the banner-based gacha game, [41] shows that the maximum cost of limited-time gacha was higher although the average probability of obtaining the rarest items was increased. Correspondingly, we show the banner-based gacha game with the succeed-after-opt-out mechanism is friendly to the buyers and can achieve a higher seller's revenue when the buyers are budget-constrained. Besides, [42] discusses the revenue-generating mechanism in gacha game, where players may be given free or discounted gacha pulls, but have to pay to get more. We model this mechanism as subsidies in gacha game and discuss its effect. In many F2P gacha games, the player may accumulate the free gacha pulls through grinding [31, 45]. We theoretically model the buyer's grinding behavior caused by subsidies.

In the operation management literature, our work connects with the dual streams of papers on opaque selling [4, 14, 19, 20]. Gacha games are an example of opaque selling where the buyers can only buy the gacha pull and each gacha pull provides a certain probability for the buyers to win the gacha reward. Recent works [7, 8, 12] have focused on selling with lottery to manage imbalanced customer demand or induce opportunities for price discrimination. Our gacha game framework diverges from the standard lottery selling in a number of key ways. First, the probability to win the gacha game in each gacha pull is varying during the buyer's gacha pulling process. The simple lottery selling is one of the special cases of the gacha game, i,e., the fixed-probability gacha game. Second, we model the complex interactions between the gacha seller and the buyer, where the sequential decision of the buyer is modeled as a Markov Decision Process (MDP), as opposed to prior work which has focused on the one-shot behavior of the buyers. Third, we investigate several interesting characteristics and mechanisms in gacha games, such as whale property, varying-probability mechanism, reset-after-winning mechanism, multi-item gacha game design, etc, which to the best of our knowledge, is the first work to provide theoretical analysis for these mechanisms.

Our work also resembles and references the work on loot box [11]. Loot box is a consumable item with a random allocation of several items, whose contents are not revealed until after purchase. The authors in [11] focus on two types of loot boxes, namely, the traditional box and the unique box, which are classified based on whether the loot box contains duplicate items. Distinguished from the previous work on loot box, we consider several interesting characteristics and mechanisms in gacha games, which are typical and fundamental mechanisms in the gacha games, but may not be applied to the classical loot box in [11].

Besides, our work distinguishes from [15]. The author in [15] investigates the optimal pricing problem in gacha game, where the consumers behave according to prospect theory and present preference for gambling. Whereas the consumers in our work are risk-neutral. Besides, we also investigate several common mechanisms and phenomena in the gacha game, which to the best of our knowledge, have not been mathematically studied yet.

Moreover, our work contributes to the emerging literature on operation management in video games. In [21] , the authors investigate the problem of maximizing players' engagement in online video games. [40] studies the incentivized actions in freemium games. [23] compares the transparent selling strategy and the opaque selling strategy in the free-to-play games. [49] proposes a dynamic model of the player's level-progression decision in online gaming. [17] considers the problem of designing video games so that players with different resources play diverse strategies. [48] studies the effect of random reward mechanisms in video games on player experience. Our work is the first to provide a systematic analysis for the gacha game.

Besides, our work can be widely applied to many scenarios beyond the video game, such as license plate lottery and probabilistic selling [30, 39, 46]. Specifically, we provide a detailed case study for the application of the gacha game on the blockchain system. To the best of our knowledge, this is the first gacha game model for the blockchain system. Some of the existing works model the Proof-of-Stake (PoS) blockchain as a lottery. [22] models the PoS protocol as a lottery and provides a fairness analysis for blockchain incentives. [6] investigates the anonymous lottery in the PoS setting. [28] designs a block producing algorithm based on PoS named Proof-of-Lottery to achieve high scalability. Compared to the existing work, we first model the blockchain system as the gacha game, and adopt the mechanisms in gacha game to analyze the existing protocols.

3 GACHA GAME MODEL OVERVIEW

We consider a revenue-maximizing seller selling a specific item using the gacha game framework. A buyer's valuation for the reward of the gacha game is described by the non-negative variable R, where R is drawn from a distribution F. The mean and variance of F are denoted by μ and σ^2, respectively. We assume that the seller knows the distribution F. This is a common assumption when we need to maximize the seller's revenue [18, 33]. Besides, it is also practical in some scenarios such as video games [10] and online platforms [16]. In the gacha game, the seller will first design the gacha game configuration, including the probability of winning the gacha game in the i-th gacha pull p_i and the price of each gacha pull c. The buyer will then choose whether to buy the gacha pull after knowing the gacha game configuration, which can be modeled as a Markov Decision Process (MDP). The interaction between the buyer and the seller can be modeled as a two-stage Stackelberg game.

3.1 Buyer Stage

The buyer will decide whether to buy the gacha pull based on the game configuration and his personal valuation R of the item in the gacha game. The buyer's utility is defined as the valuation of the item in the gacha game minus the cost for buying the gacha pulls. The buyer behavior can be modeled as a Markov Decision Process (MDP) represented as a 4-tuple $(\mathcal{S}, \mathcal{A}, \mathcal{P}_a, \mathcal{R}_a)$, which is shown in Figure 1. $\mathcal{S} = \{S_1, S_2, S_3, \cdots\}$ is a set of states, which can be either finite or infinite. $S_i \in \mathcal{S}$ denotes the state that the buyer has pulled $i-1$ gacha pulls but does not win the desired item, that is, the buyer is currently at the i-th round of the gacha game. Specially, $S^* \in \mathcal{S}$ denotes

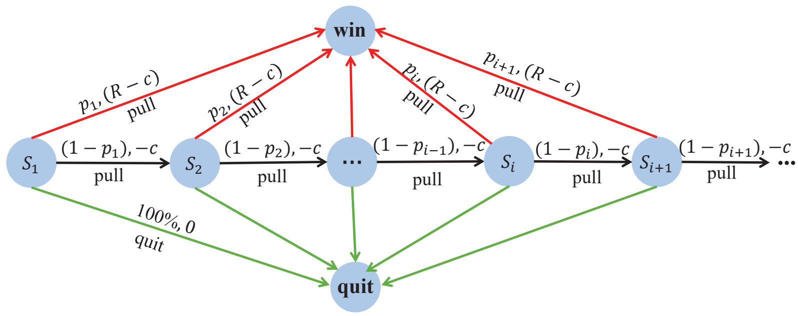


Fig. 1. Markov Decision Process in Gacha Game

the absorbing terminal state that the buyer wins the gacha game. $S^q \in \mathcal{S}$ denotes the terminal state that the buyer quits the gacha game. Note that we consider each gacha pull as a round in the gacha game. $\mathcal{A}$ is the action space of the MDP. There are only two actions in each state of this MDP, namely, a_p and a_q. The action a_p means that the buyer will buy a gacha pull, and the action a_q means that the buyer quits the gacha game. $P_a(S,S') = \Pr(S_{t+1} = S' | S_t = S, a_t = a)$ is the transition probability that action a in state S at time t will lead to state S' at time $t+1$. Specifically, we have

$$P_{a_p}(S_i, S_{i+1}) = 1 - p_i; \quad P_{a_p}(S_i, S^*) = p_i; \quad P_{a_q}(S_i, S^q) = 1.$$

$R_a(S,S')$ is the expected immediate reward received after transitioning from state S to state S'. Specifically, we have

$$R_{a_p}(S_i, S_{i+1}) = -c; \quad R_{a_p}(S_i, S^*) = R - c; \quad R_{a_q}(S_i, S^q) = 0.$$

The buyer is utility-maximizing and will pick up the optimal policy π^* to maximize his utility. The buyer behavior is indeed a sequential decision, and the buyer only needs to decide when to stop pulling the gacha game. Therefore, the set of the available policies in this MDP is $\{\pi_0, \pi_1, \pi_2, \cdots, \pi_\infty\}$. Here π_k denotes the policy that the buyer will pull the gacha until he wins the gacha game midway or he has reached the state S_{k+1} and then stops pulling. Specially, π_∞ denotes the policy that the buyer will always pull the gacha until he wins the gacha game, and π_0 denotes the policy that the buyer will never pull the gacha. The value of MDP of the policy π_k at state S_i is

$$V_{\pi_k}(S_i) = \begin{cases} 0, & k < i, \\ p_k R - c, & k = i, \\ p_i R - c + (1-p_i)\gamma V_{\pi_k}(S_{i+1}), & k > i, \end{cases}$$

where γ is the discount factor satisfying $0 \le \gamma \le 1$ in MDP. In our gacha game model, we assume that $\gamma = 1$, which is reasonable because $V_{\pi_k}(S_i)$ is bounded, and practically, a gacha game can be considered as a short-term procedure, where the discount effect is not significant. Then the value of MDP of the policy π_k at state S_i is formulated in the following lemma.

Lemma 3.1. *The value of MDP of the policy π_k at state S_i is*

$$V_{\pi_k}(S_i) = \left(1 - \prod_{j=i}^{k}(1-p_j)\right) R - \left(p_i + \sum_{m=i+1}^{k}(m-i+1)p_m \prod_{j=i}^{m-1}(1-p_j) + (k-i+1)\prod_{j=i}^{k}(1-p_j)\right) c.$$

The buyer in the gacha game is rational. Initially, the buyer is at state S_1, and will pull the gacha game if and only if $\exists k > 0, V_{\pi_k}(S_1) \ge 0$. The optimal policy for the buyer at state S_i is investigated in the following lemma.

Lemma 3.2. *The policy π_k at state S_i is optimal if and only if the following conditions are satisfied:*

(1) $V_{\pi_k}(S_j) \geq 0, \quad \forall j \in [i, k]$.
(2) $V_{\pi_l}(S_{k+1}) \leq 0, \quad \forall l \in [k+1, \infty)$.

Lemma 3.2 implies that the optimal policy in the gacha game is deterministic. Specifically, if π_k is the optimal policy at state S_i, then π_k is also the optimal policy for $S_j,\ \forall j \in [i, k]$. Specially, the optimal for the buyer at the initial state S_1 is π_k, where $k = \arg\max_l V_{\pi_l}(S_1)$. And this policy would not be changed during the gacha pulling process.

3.2 Seller Stage

We assume that the seller is a monopolist and wants to maximize his revenue. The seller will first design the gacha game configuration including the probability of winning the gacha game in the i-th gacha pull p_i and the price of each a gacha pull c. The revenue of the seller comes from selling the gacha pulls to the buyers. For a buyer with policy π_k, the expected number of gacha pulls that he will buy is $E(\pi_k)$, which is formulated in the following lemma. Then the expected revenue of the seller obtaining from selling the gacha pulls to this buyer is $c \cdot E(\pi_k)$. Note that the buyer's policy π_k is highly dependent on the game configuration. Therefore, to maximize the seller's revenue, the seller should carefully design the game configuration.

Lemma 3.3. *For the buyer who adopts policy π_k and is currently at state S_i, the expected number of gacha pulls that the buyer will buy is*

$$E(\pi_k, S_i) = \begin{cases} 0, & k < i, \\ p_i + \sum_{j=i+1}^{k} (j - i + 1) p_j \prod_{l=i}^{j-1} (1 - p_l) + (k - i + 1) \prod_{j=i}^{k} (1 - p_j), & k \geq i. \end{cases}$$

Briefly, when there is no further confusion, we denote the expected number of gacha pulls that a buyer with policy π_k will buy at initial state S_1 as $E(\pi_k) = E(\pi_k, S_1)$. Besides, the expected number of gacha pulls needed to win the gacha game is denoted as $E(\pi_\infty) = E(\pi_\infty, S_1)$.

3.3 Typical Gacha Game

We divide the gacha games into two categories as follows:

Fixed-Probability Gacha Game: A fixed-probability gacha game is a gacha game where the probability of winning the gacha game remains the same during the buyer's gacha pull process, i.e., $p_i \equiv p$, where p is a fixed probability. Specially, the lottery is indeed a fixed-probability gacha game, where the ticket in the lottery corresponds to the gacha pull in the gacha game.

Varying-Probability Gacha Game: A varying-probability gacha game is a gacha game where the probability of winning the gacha game in each gacha pull may be different, i.e., we allow that $\exists i \neq j, p_i \neq p_j$. The fixed-probability gacha game is just a special case of the varying-probability gacha game. Specially, the winning guarantee mechanism in the varying-probability gacha game, i.e., $\exists N,$ s.t. $P_N = 1$, is called "pity system" [50], which is widely adopted to prevent the aggravating situation where the buyer buys numerous gacha pulls but receives nothing.

4 REVENUE OPTIMAL SINGLE-ITEM GACHA GAME

In this section, we first introduce the definition of "Whale Property", and then show that the whale property gacha game can achieve the maximum seller's revenue, while the non-whale property gacha game performs worse. We also show the equivalence of the gacha game and the single-item single-bidder Myerson auction and further figure out the optimal gacha game configuration that can achieve the maximum seller's revenue. Besides, when the buyer has budget constraints, the gacha game can achieve a higher seller's revenue compared to the "take-it-or-leave-it" strategy.

4.1 Whale Property

In many online video games with gacha mechanisms, once the buyer pulls one gacha pull, he is likely to continue pulling gacha until he wins the gacha game [44]. In this paper, we define this addictive property as "whale property" for the rational buyer, where the definition comes from the fact that the high-spending players are often colloquially referred to as "whales" [47]. For the gacha game with the whale property, the optimal policy for a rational buyer is to either never pull the gacha game, i.e., adopt the policy π_0, or continue pulling the gacha game until he wins the gacha game, i.e., adopt the policy π_∞. The whale property is mathematically defined as follows.

Definition 4.1. (**Whale Property**) A gacha game $\mathbb{G}$ has the whale property if and only if for any buyer with any non-negative valuation in the gacha game, his optimal policy is either π_0 or π_∞, that is, the following condition always holds:

$$\left(\exists k, V_{\pi_k}(S_1) \geq 0\right) \Rightarrow \left(\forall i, V_{\pi_\infty}(S_i) \geq 0\right),$$

i.e., if it is profitable for the buyer to pull the gacha game initially, he will continue pulling the gacha game until he wins, otherwise, he will never pull.

For the gacha game with the whale property, the optimal policy for the buyer will be either π_0 or π_∞, i.e., ("take-it-or-leave-it"), which is shown in the following lemma.

Lemma 4.2. *The optimal policy π^* for the buyer in the gacha game with the whale property is*

$$\pi^* = \begin{cases} \pi_\infty, & \textit{When } V_{\pi_\infty}(S_1) \geq 0 \Leftrightarrow R \geq E(\pi_\infty) \cdot c, \\ \pi_0, & \textit{When } V_{\pi_\infty}(S_1) < 0 \Leftrightarrow R < E(\pi_\infty) \cdot c. \end{cases}$$

Lemma 4.2 implies that the buyer's behavior in the whale property gacha game is only related to the expected number of gacha pulls to win the gacha game $E(\pi_\infty)$ and the price for each gacha pull c, which can help to simplify the analysis of the whale property gacha game. In the whale property gacha game, if the buyer's valuation is high enough (i.e., $R \geq E(\pi_\infty) \cdot c$), he will continue pulling the gacha game until he gets the desired item. Otherwise, he will never pull the gacha game.

In some gacha games, to attract the buyers to buy the gacha pull, the seller will monotonically increase the probability of winning the gacha in each gacha pull. The following lemma shows that these gacha games with increasing probability have the whale property.

Lemma 4.3. *If the probability of winning the gacha game is monotonically increasing for each gacha pull, i.e., $p_i \leq p_{i+1}, \forall i$, the gacha game has the whale property.*

Lemma 4.3 implies that the fixed-probability gacha game and the varying-probability gacha game with non-decreasing probability have the whale property. In fact, the whale property is present in a lot of gacha games in the real world, which keeps players addicted to them [47].

For the non-whale property gacha game, the expected value in the gacha game at the initial states is high, which can attract the buyer to join in the game. And the buyer may stop pulling the gacha midway when his expected value at that state is negative. Some video games apply the novice bonus such as a high probability to win the gacha game at the beginning, or discounted price to attract new players. This novice bonus reflects the non-whale property of the gacha game.

4.2 Optimality of the Whale Property Gacha Game

In this section, we will prove the optimality of the whale property gacha game, that is, the optimal gacha game that can achieve the maximum seller's revenue must be a whale property gacha game.

Figuring out the optimal design of the gacha game is difficult due to its complexity, especially in designing the probability of each gacha pull. Fortunately, in Lemma 4.2, we have shown that in the whale property gacha game, the buyer's behavior is only related to the expected number of gacha pulls needed to win the gacha game with policy π_∞, which can help us substantially simplify the analysis of the gacha game. Therefore, for each whale property gacha game, we can represent it by a fixed-probability gacha game with the same expected number of gacha pulls to win. And we denote the winning probability of this equivalent fixed-probability gacha game as "equivalent probability", which is defined as follows.

Definition 4.4. (**Equivalent Probability**) The equivalent probability of a whale property gacha game is defined as the $\tilde{p} = 1/E(\pi_\infty)$, where $E(\pi_\infty)$ is the expected number of gacha pulls needed to winning the gacha game with policy π_∞.

With the definition of the equivalent probability, the value in MDP at the initial state S_1 for a buyer with valuation R and policy π_∞ can be simplified as $V_{\pi_\infty}(S_1) = R - c/\tilde{p}$. This implies that for the whale property gacha game, the buyer will pull the gacha game if and only if the equivalent probability of the game is large compared with his valuation, i.e., $\tilde{p} \geq c/R$.

To figure out the optimal gacha game configuration, we first consider the scenario where the buyer's valuation follows a discrete distribution as follows:

$$P(R = R_i) = \beta_i, \quad i = 1, 2, \cdots, M,$$

and $\sum_{i=1}^{M} \beta_i = 1$. Without loss of generality, we assume that $R_1 > R_2 > \cdots > R_M$. We will first show the optimality of the whale property gacha game with the discrete user valuation distribution and then extend to the continuous user valuation distribution. Thus, we can prove the optimality of the whale property gacha game with any user distribution.

Firstly, we have that the optimal whale property gacha game design with the discrete buyer's valuation distribution is shown as follows:

Lemma 4.5. *When the buyer's valuation follows the discrete distribution $P(R = R_i) = \beta_i, \quad i = 1, 2, \cdots, M$, the whale property gacha game with equivalent probability $\tilde{p} = \hat{p}_{i^*}$ can achieve the maximum seller's revenue $\frac{c}{\hat{p}_{i^*}} \sum_{j=i^*}^{M} \beta_j$, where*

$$i^* = \arg\max_{i \in \{1,2,\cdots,M\}} \frac{c}{\hat{p}_i} \sum_{j=i}^{M} \beta_j, \quad \text{and} \quad \hat{p}_i = \frac{c}{R_i}(c \leq R_i).$$

To investigate the general gacha game (with or without the whale property), we introduce the definition of whale property subgame.

Definition 4.6. (**Whale Property Subgame**) We define the gacha subgame $\mathbb{G}(i,k), (i \leq k)$ as the slice of the gacha game starting from buyer's state S_i ending at buyer's state S_k. Then a gacha subgame $\mathbb{G}(i,k)$ has the whale property if and only if for any buyer in the gacha game, his optimal policy in this gacha subgame is either π_{i-1} or π_k, that is, the following condition always holds:

$$\left(\exists l \in [i,k], V_{\pi_l}(S_i) \geq 0\right) \Rightarrow \left(\forall j \in [i,k], V_{\pi_k}(S_j) \geq 0\right),$$

i.e., if it is profitable for the buyer to pull the gacha subgame at state S_i, he will continue pulling the gacha subgame until he wins or reaches the state S_{k+1}, otherwise, he will never pull.

Tackling the non-whale property gacha game is difficult, because the optimal policy of the buyer is complicated and the buyer may quit midway. Luckily, we find that each gacha game can be divided into several whale property subgames. Specially, the gacha pull in each round can be considered as a whale property subgame, i.e., whether or not to pull the current round. Therefore, we can investigate the non-whale property gacha game through its whale property subgames.

As shown in Figure 2, a gacha game $\mathbb{G}$ that can be divided into L consecutive whale property subgames, namely, $\mathbb{G}(a_{i-1}+1, a_i), i = 1, 2, \cdots, L$, where $a_0 = 0$ and $a_L = \infty$. Since the subgame

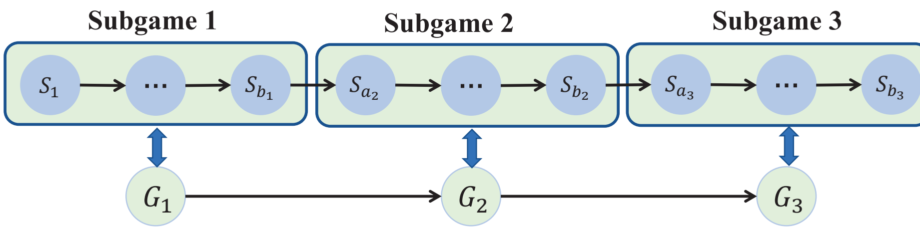


Fig. 2. The non-whale property gacha game can be divided into 3 whale property subgames, where each subgame can be represented by a fixed-probability gacha game with the corresponding equivalent probability.

$\mathbb{G}(a_{i-1}+1, a_i)$ has the whale property, we can represent the subgame by a fixed-probability gacha game with the equivalent probabilities $\tilde{p}_i$. Besides, the subgame whale property guarantees that if the buyer pulls the gacha game at the state $S_{a_{i-1}+1}$, he will continue pulling until he wins the gacha game or reaches the state S_{a_i+1}. Then, to figure out the optimal design for the general gacha game, we can turn to investigating its whale property subgames.

Lemma 4.7. *Suppose that the buyer's valuation follows the discrete distribution $P(R = R_i) = \beta_i, \quad i = 1, 2, \cdots, M$. Consider a gacha game $\mathbb{G}^*$ that can be divided $L(L \le M)$ whale property subgames, and the i-th whale property subgame contains n_i rounds of gacha pulls, i.e., $n_i = a_i - a_{i-1}, n_i \ge 0$, where $i = 1, 2, \cdots, M$. Specially, the length of i-th subgame being $n_i = 0$ implies that the i-th whale property subgame is dummy. The equivalent probability of the i-th subgame is $\tilde{p}_i = c/R_i (c \le R_i)$ and the lengths of these whale property subgames $\boldsymbol{n} = (n_1, n_2, \cdots, n_M)$ are*

$$\arg\max_{\boldsymbol{n}} c \cdot \sum_{k=1}^{M} \beta_k Q_k(\boldsymbol{n}), \quad n_i \ge 0, \forall i = 1, 2, \cdots, M,$$

where $Q_k(\boldsymbol{n})$ is the expected number of gacha pulls bought by the buyer with valuation R_k in the gacha game with the lengths of the whale property subgames $\boldsymbol{n}$, which is formulated as follows:

$$Q_k(\boldsymbol{n}) = \begin{cases} \frac{1-(1-\tilde{p}_1)^{n_1}}{\tilde{p}_1}, & k = 1, \\ \left(\sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} (1-\tilde{p}_j)^{n_j}\right) \frac{1-(1-\tilde{p}_i)^{n}_i}{\tilde{p}_i}\right), & k > 1. \end{cases}$$

Then the gacha game $\mathbb{G}^*$ is optimal and can achieve the maximum seller's revenue.

According to Lemma 4.7, we can find out the optimal design of the gacha game by turning it into an optimization problem. And the solution of the optimization problem implies that the gacha game $\mathbb{G}^*$ has the whale property. Suppose that there exists a gacha game $\mathbb{G}'$ with $L(L > 1)$ whale property subgames which does not have the whale property but is also optimal and can achieve the maximum seller's revenue. We will show that the seller's revenue in $\mathbb{G}^*$ in Lemma 4.7 is greater than that in $\mathbb{G}'$, which will lead to the contradiction and thus prove the optimality of the whale property gacha game. And the main result is shown in the following theorem, which still holds when the buyer's valuation follows the arbitrary distribution.

Theorem 4.8. *The optimal gacha game that can achieve maximum seller's revenue must be a whale property gacha game, that is, the maximum seller's revenue of the non-whale property gacha game is strictly less than that of the whale property gacha game.*

Compared to the whale property gacha game, the non-whale property gacha game can encourage the buyers who have low valuation and would not conduct any gacha pull in the optimal whale property gacha game to pull the gacha instead of leaving directly. Thus the non-whale property helps to expand the number of participants in the gacha game. Although the non-whale property offers advantages in terms of more participants, the buyers who will pull in the optimal whale property gacha game spend less money on gacha pulling, leading to a lower seller's revenue. Indeed, many gacha games have whale property. A recent research [26] shows that the high-spending players in the gacha game essentially subsidize the game for other players who may spend smaller amounts of money, or even spend no money at all, which is consistent with the theoretical result of the whale property gacha game.

4.3 Equivalence of Gacha Game and Single-item Single-bidder Auction

In this section, we show the equivalence of the gacha game and the single-item single-bidder auction [33] and further figure out the optimal gacha game design that can achieve the maximum seller's revenue. Such an equivalence reveals the corresponding revenue structure between gacha game and auction.

Let's recall the single-item single-bidder auction [33], where there is a bidder with his personal valuation R on the item in the auction and a revenue-maximum auctioneer. In the auction mechanism, the auctioneer will first design an allocation rule $x(b) \in [0, 1]$ and the payment rule $y(b) \in \mathbb{R}$, and then a rational bidder will propose his bid $b \in \mathbb{R}$. When the bidder proposes a bid $b \in \mathbb{R}$, he needs to pay $y(b)$ and can get the item in the auction with probability $x(b)$. Thus, the utility of the bidder with bid b is $u(b) = R \cdot x(b) - y(b)$. The following theorem shows the equivalence of the gacha game and the single-item single-bidder Myerson auction with a stochastic allocation rule.

Theorem 4.9. *Consider a gacha game $\mathbb{G}$, which can be divided into L consecutive whale property subgames, namely, $\mathbb{G}(a_{i-1} + 1, a_i), i = 1, 2, \cdots, L$, where $a_0 = 0$ and $a_L = \infty$. The gacha game $\mathbb{G}$ is equivalent to the single-item single-bidder Myerson auction with the allocation rule $x(b) = P_{succ}(\pi_{opt(b)})$ and the payment rule $y(b) = E(\pi_{opt(b)}) \cdot c$, where b is the bidding value in auction, which is equivalent to the bidder's valuation R due to the dominant-strategy incentive-compatible (DSIC) property, $\pi_{opt(R)}$ denotes the optimal gacha pulling policy for a buyer with valuation R at initial state S_1, which is formulated as*

$$opt(R) = \begin{cases} a_0 = 0, & R \le \frac{E(\pi_{a_1}) \cdot c}{P_{succ}(\pi_{a_1})}, \\ a_i, & \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{succ}(\pi_{a_i}) - P_{succ}(\pi_{a_{i-1}})} < R \le \frac{\left(E(\pi_{a_{i+1}}) - E(\pi_{a_i})\right) \cdot c}{P_{succ}(\pi_{a_{i+1}}) - P_{succ}(\pi_{a_i})}, \\ a_L = \infty, & R > \frac{\left(E(\pi_\infty) - E(\pi_{a_{L-1}})\right) \cdot c}{P_{succ}(\pi_\infty) - P_{succ}(\pi_{a_{L-1}})}, \end{cases}$$

$P_{succ}(\pi_k) = 1 - \prod_{j=1}^{k}(1 - p_j)$ denotes the probability of winning the gacha game with policy π_k, and $E(\pi_k)$ denotes the expected number of gacha pulls with policy π_k, which is formulated in Lemma 3.3.

The auctioneer and the bidder in the auction correspond to the seller and the buyer in the gacha game, respectively. According to Myerson's Lemma [33], the auction mechanism in Theorem 4.9 is dominant-strategy incentive-compatible (DSIC), and can guarantee that the bidder will report truthfully, i.e., $b \equiv R$. The buyer with his personal valuation R will pull the gacha game with his optimal policy $\pi_{\text{opt}(R)}$, which can provide the buyer with the probability $P_{\text{succ}}(\pi_{\text{opt}(R)})$ to win the gacha game at the expected cost of $E(\pi_{\text{opt}(R)}) \cdot c$. Correspondingly, the truthful bidder with valuation R will propose a bid $b = R$, and get the item with the probability $x(b)$ and payment $y(b)$. The winning probability and the expected cost in the gacha game correspond to the allocation rule and the payment rule in the auction, respectively.

With the bidder/buyer's valuation drawn from distribution F, according to [33], the optimal allocation rule and the payment rule in the single-item single-bidder Myerson auction that can achieve the maximum revenue are $x(b) = \begin{cases} 0, & b \le r^*, \\ 1, & b > r^*, \end{cases}$ $y(b) = \begin{cases} 0, & b \le r^*, \\ r, & b > r^*, \end{cases}$ where $r^* = \arg\max_r r \cdot (1 - F(r))$ is the reserved price set by the auctioneer/seller. This optimal auction mechanism corresponds to the gacha game with $L = 1$ whale property subgame in Theorem 4.9. This implies the optimality of the whale property gacha game, which is consistent with Theorem 4.8. And the optimal gacha game design is shown in the following theorem.

THEOREM 4.10. *The optimal gacha game that can achieve the maximum seller's revenue should have whale property and satisfy the following condition:*

$$E(\pi_\infty) \cdot c = r^* = \arg\max_r r \cdot (1 - F(r)),$$

where c is the cost of each gacha pull, $E(\pi_\infty)$ is the expected number of gacha pulls to win the gacha game with policy π_∞, r^* is the optimal reserved price in the single-item single-bidder Myerson auction.

Theorem 4.10 implies that the maximum seller's revenue in the gacha game is equivalent to that in the single-item single-bidder Myerson auction. The optimality of the whale property gacha game revealed in Theorem 4.8 and Theorem 4.10 is consistent with the optimality of the "take-it-or-leave-it" (non-haggling) strategy on one good [37]. This finding provides a practical insight for the gacha game design and explains the popularity of whale property design in many gacha games. Besides, we will show that with budget constraints, the gacha game can achieve a higher seller's revenue than the "take-it-or-leave-it" selling strategy.

For the "take-it-or-leave-it" strategy, a buyer will buy the item only when the price of the item does not exceed his valuation and his budget. While in the gacha game, the buyer with a high valuation will continue purchasing the gacha pulls until he either wins the gacha game or exhausts his budget. Therefore, selling in the gacha game is quite robust to random fluctuations in buyers' budgets. Specifically, for a buyer with budget B, he can buy at most $\lfloor \frac{B}{c} \rfloor$ gacha pulls, where c is the price for each gacha pull. Thus, the buyer's policy can only be $\pi_i, i = 0, 1, \cdots, \lfloor \frac{B}{c} \rfloor$. Then a buyer with budget B will pull the gacha game if and only if $\exists 0 < k \leq \frac{B}{c}, V_{\pi_k}(S_1) \geq 0$.

PROPOSITION 1. *With budget constraints, the whale property gacha game can achieve a higher seller's revenue than the "take-it-or-leave-it" strategy.*

The following example demonstrates the advantage of the gacha game when facing budget-constrained buyers. The detailed calculation of the following example is shown in Appendix E

Example 1. There is a buyer who has the valuation of 100 and his budget B follows the distribution that $P(B = 50) = 0.5, P(B = 100) = 0.5$. The maximum seller's revenue achieved by the "take-it-or-leave-it" strategy is 50, whereas the fixed-probability gacha game with the probability being 0.01 and the price for each gacha pull being 1, can achieve the seller's revenue of 51.448. This is because the buyer in the gacha game will continue buying the gacha pulls until he either wins the game or exhausts his budget, while in the "take-it-or-leave-it" selling, only the buyer with a larger budget than the selling price will buy the item. Therefore, the gacha game can help the seller to achieve a higher revenue from the buyers with small budgets.

5 MULTI-ITEM GACHA GAME

In this section, we mainly focus on the whale property gacha game due to its optimality and popularity, and explore the multi-item gacha game, which includes multiple phases and each phase contains exactly one item. The buyer can play the gacha game in each phase. Once the buyer wins the gacha game, he can get the corresponding item and reward in that phase. We mainly focus on two popular and common types of the multi-item gacha games, namely, the sequential multi-item gacha game and the banner-based multi-item gacha game, which are classified based on how their phase ends. The sequential multi-item gacha game will end a phase and enter the next phase only when the buyer wins the gacha game once. The banner-based multi-item gacha game allows buyer's opt-out, and will end a phase and enter the next phase when the buyer wins the gacha game once or chooses to opt-out. The flexibility of the opt-out provision makes the banner-based gacha game distinguished from the sequential gacha game.

5.1 Sequential Multi-Item Gacha Game

Sequential multi-item gacha game is widely adopted in many online video games, which can encourage the players to pull and win the gacha game more than once. In this section, we consider a multi-item sequential gacha game with K items (K phases), where the buyer's valuation for the k-th item in the k-th phase is $R_k (k \leq K)$. When the buyer wins the gacha game in the k-th phase, he will obtain the corresponding reward R_k and enter the next phase, i.e., $(k+1)$-th phase $(k < K)$ or the game ends $(k = K)$. That is, the buyer will sequentially get the reward $R_1, R_2, \cdots, R_k$ if he has won the gacha game k times $(k \leq K)$.

In this section, we mainly focus on the reset-after-winning mechanism and the succeed-after-winning mechanism in the sequential multi-item gacha game. For the sequential gacha game with the reset-after-winning mechanism, the buyer's state in the gacha game will be reset whenever the buyer wins the gacha game and enters the next phase. Specifically, if a buyer wins the gacha game at state S_k, the reset-after-winning mechanism will reset the buyer's state, i.e., the buyer will be at state S_1 in the next phase. As an opposite mechanism, the succeed-after-winning mechanism will not reset the buyer's state. Specifically, if a buyer wins the gacha game at state S_k, he will be at state S_{k+1} in the next phase.

We now investigate the buyer's behavior in the whale property sequential multi-item gacha game with the reset-after-winning mechanism, which is shown in the following proposition.

Proposition 2. *For the whale property sequential multi-item gacha game with the reset-after-winning mechanism, the buyer will continue pulling the gacha until he has won k^* times, where*

$$k^* = \arg \max_{k=0,1,2,\cdots,K} \left\{ \left(\sum_{j=1}^{k} R_j \right) - kE(\pi_\infty) \cdot c \right\}. \tag{1}$$

Specially, $k^* = 0$ implies that the buyer will never pull the gacha game. Here $E(\pi_\infty)$ denotes the expected number of gacha pulls needed to win the gacha game once, which is formulated in Lemma 3.3, and c denotes the price of each gacha pull.

We next investigate the buyer's behavior in the whale property sequential gacha game with the succeed-after-winning mechanism. Compared to the gacha game with the reset-after-winning mechanism, the buyer's behavior in the gacha game with the succeed-after-winning mechanism is more complicated. This is due to the fact that the expected cost for every win in the gacha game varies and is greatly dependent on the buyer's current state. As a result, throughout the gacha pulling process, the buyer will make a dynamic choice, as illustrated by the following proposition.

Proposition 3. *For the whale property sequential multi-item gacha game with the succeed-after-winning mechanism, the buyer that has won the gacha game k times ($k = 0, 1, \cdots, K-1$), will pull the gacha at state S_i if and only if*

$$\max_{t=1,2,\cdots,K-k} \left\{ \left(\sum_{j=k+1}^{k+t} R_j \right) - H(t,i) \cdot c \right\} \geq 0,$$

where $H(t,i)$ denotes the expected number of gacha pulls needed to win the gacha game t more times when the buyer is at state S_i, which can be recursively calculated as follows:

$$H(t,i) = \begin{cases} \sum_{j=i}^{\infty} p_j \prod_{t=i}^{j-1} (1-p_t) \cdot (j-i+1), & t = 1, \\ \sum_{j=i}^{\infty} p_j \prod_{t=i}^{j-1} (1-p_t) \cdot (j-i+1+H(t-1,j+1)), & t > 1, \end{cases} \tag{2}$$

where p_i is the probability to win the gacha game at state S_i.

Intuitively, in the gacha game with non-decreasing probability, resetting the buyer's state whenever the buyer wins the gacha game can help the seller to achieve a higher revenue because the buyers need to take more gacha pulls to win the game again. While the succeed-after-winning mechanism seems to benefit the buyer but harm the seller's revenue because the buyer can take fewer gacha pulls to win the gacha game after he has won the gacha game. Indeed, many online gacha games with non-decreasing probability and pity system such as "Genshin Impact" adopt the reset-after-winning mechanism. However, we present the following counterintuitive result.

INSIGHT 1. *The succeed-after-winning mechanism can achieve a higher seller's revenue compared to the reset-after-winning mechanism in the sequential multi-item gacha game.*

Here we show an example to support the counterintuitive finding above.

Example 2. Suppose that there are two items in the sequential gacha game, i.e., $K = 2$. The buyer's valuations for these two items are independently and identically distributed (i.i.d.), and follow the uniform distribution $[0, 1]$. By separately selling these two items at the same price, the maximum seller's revenue is 0.5. With the reset-after-winning mechanism, the maximum seller's revenue that the sequential gacha game can achieve is 0.516. While for the sequential gacha game with the succeed-after-winning mechanism and pity system where $N = 100$, $p_i = 0.172, \forall i < N$ and $P_N = 1$ and the price of the gacha pull $c = 0.01$, the seller's revenue is 0.5218.

Reasoning. With the reset-after-winning mechanism, the expected cost for each win of the gacha game remains the same. While with the succeed-after-winning mechanism, the expected cost for each win of the gacha game varies. The varying cost makes it possible to achieve a higher seller's revenue. In Example 2, the expected cost for winning the second item is lower than the expected cost for winning the first item due to the pity system. The lower cost of the second item attracts the buyer to pull the gacha game, which makes it possible to increase the seller's revenue.

We next investigate a special asymptotic scenario, where there are infinite number of items, and the buyer's valuation of each item is i.i.d. random variable with mean μ and variance σ^2.

THEOREM 5.1. *For the sequential multi-item gacha game with infinite items, and the buyer's valuation of each item follows the identical and independent distribution with mean μ and variance σ^2, the whale property sequential multi-item gacha game with the reset-after-winning mechanism can achieve the asymptotic optimality on seller's revenue, i.e.,*

$$\lim_{K\to\infty} \frac{c \cdot \mathbb{E}(\#\ of\ gacha\ pulls\ purchases)}{K} = \mu,$$

where $\frac{c\cdot\mathbb{E}(\#\ of\ gacha\ pulls\ purchases)}{K}$ denotes the normalized seller's revenue, c is the price of each gacha pull and K is the number of items in the gacha game.

Theorem 5.1 shows that the reset-after-winning mechanism is asymptotically revenue-optimal, which implies that the reset-after-winning mechanism in the sequential multi-item gacha game can achieve a satisfactory seller's revenue. This explains the popularity and widespread use of the reset-after-winning mechanism in the gacha game. Besides, compared to the bundle selling [5], which has been shown to be asymptotically revenue-optimal, the gacha game is friendlier to the buyers that prefer smaller purchases or have a smaller budget, and can achieve a higher seller's revenue when considering buyer's budget. This is because the bundle strategy requires the buyer either to purchase all of the items or none of them, whereas the gacha game even allows the buyer to buy just one gacha pull.

5.2 Banner-based Multi-Item Gacha Game

In this section, we explore another multi-item gacha game named banner-based multi-item gacha game. Due to the asymptotic optimality and popularity of the reset-after-winning mechanism, we assume that the banner-based gacha game discussed in this paper adopts the reset-after-winning mechanism and the buyer's state will be reset whenever he wins.

The only difference between banner-based and previous sequential gacha game is that banner-based gacha game allows the buyer's opt-out in some banners (phases). Therefore, the buyer can choose to pull in some of the banners without the consecutive requirement. The banner-based multi-item gacha game is also widely adopted in many online video games such as "Genshin Impact". In these banner-based gacha games, specific items can only be acquired during a specific period of time. This event time period is referred to as a "banner" and each banner refers to a phase in the multi-item gacha game. Each banner contains a specific item and can be considered as a single-item gacha game. After a period of time, the next banner will replace the current one. Compared to other probabilistic selling mechanisms, such as loot box, the banner-based gacha game is more suitable for adding new items. Besides, empirical results [41] show that this limited-time gacha can contribute to the seller's revenue growth. This explains why banner-based gacha game design gains increasing popularity in a variety of online video games.

Here, we consider a banner-based gacha game with K banners, where the buyer's valuation of the item in the k-th banner is denoted as R_k. We focus on the reset-after-opt-out mechanism and the succeed-after-opt-out mechanism in the banner-based gacha game. With the reset-after-opt-out mechanism, the buyer's state will be reset at the beginning of the next banner when the buyer chooses to opt-out. Specifically, for a buyer that has pulled i gacha pulls and is currently at state S_{i+1}, if he stops pulling in the current banner and chooses to opt-out for the next banner, he will be at state S_1 in the next banner. As an opposite mechanism, the succeed-after-opt-out mechanism will carry the buyer's state from one banner to another, that is, for a buyer that is currently at state S_{i+1}, if he stops pulling in the current banners and chooses to opt-out for the next banner, he will be at state S_{i+1} in the next banner.

We next investigate the relation among the reset-after-opt-out mechanism and the succeed-after-opt-out mechanism in the banner-based gacha game, and the separate selling with several independent single-item gacha games. We can find that the banner-based gacha game with the reset-after-opt-out mechanism is equivalent to the separate selling with several independent single-item gacha games, because the buyer's utility and behavior in one banner will not affect those in another banner. However, in the banner-based gacha game with the succeed-after-opt-out mechanism, the buyer may adopt more sophisticated gacha pulling strategies to maximize his utility. For instance, the buyer may consider pulling some gacha pulls in one banner and then stop, carrying his state to the next banner. The following theorem shows the equivalence of the banner-based gacha game with the reset-after-opt-out mechanism, the banner-based gacha game with the succeed-after-opt-out mechanism, and the separate selling with several independent single-item gacha games.

THEOREM 5.2. *The banner-based gacha game with the reset-after-opt-out mechanism, the banner-based gacha game with the succeed-after-opt-out mechanism, and the separate selling with several independent single-item gacha games are equivalent, i.e., the behaviors of the rational buyers and the seller's revenues in these gacha games are the same.*

Theorem 5.2 shows the interesting results that the banner-based gacha game works like the several independent gacha games. We can find that the succeed-after-opt-out mechanism in the banner-based multi-item gacha game is friendly to the buyers since the buyers can quit midway in the current banner and remain their states to the banner that they want to pull. In the gacha game with non-decreasing probability, this inherited state can help the buyers to win the gacha game with fewer gacha pulls. Besides, when considering buyer's time-varying budget constraint, the succeed-after-opt-out mechanism in the banner-based gacha game can help the seller to achieve a higher revenue. Here we show an example to support this claim.

Example 3. Consider that buyer is budget-constrained and get some periodical income $I = 50$ in the time frame of each banner, such as monthly salary. There are two banners in this game and the buyer's valuation of the reward in these banners are $R_1 = 100, R_2 = 50$. The price for each gacha pull is $c = 1$. Consider the banner-based gacha game where $N = 100$, $p_i = 0.01, \forall i < N$ and $p_N = 1$.

- With the reset-after-opt-out mechanism, the buyer will pull in the first banner and will never pull in the second banner. In this case, the seller's expected revenue is 39.499.
- With the succeed-after-opt-out mechanism, the buyer will first pull in the first banner. If the buyer exhausts his budget but fails to win in the first banner, the buyer's state will be inherited to the second banner, which will lower the cost to win in the second banner. Therefore, the buyer will pull in the second banner. In this case, the seller's expected revenue is 63.397.

Theorem 5.2 and the discussion above demonstrate the efficiency of the succeed-after-opt-out mechanism in banner-based multi-item gacha game, and provide a theoretical justification for its widespread use in the online video games.

6 SUBSIDIES AND GRINDING BEHAVIOR IN GACHA GAME

In this section, we investigate the subsidies in the gacha game, where the seller will provide free gacha pulls to the buyer as subsidies. This is a common practice in many online video games [42]. Specially, players in online video games can obtain some free gacha pulls by finishing some commissions, which motivates the players to play the game. The additional subsidies allow the seller to further steer buyer's behavior, especially for the scenarios where the game configuration is not frequently changed.

Note that subsidies in the gacha game are different from the traditional subsidization where the subsidy serves as a price discount [13]. With free gacha pulls, the buyers may get the reward of the gacha game for free without buying any gacha pulls from the seller. This property makes the subsidies in the gacha game more complicated and may result in buyer's grinding behavior.

6.1 Subsidies in Single-Item Gacha Game

We first explore the subsidies in the single-item gacha game, which can demonstrate how the subsidies affect both the buyer's behavior and the seller's revenue.

Suppose that buyer is subsidized with m free gacha pulls at the beginning of the gacha game. If the buyer does not win the gacha game within m gacha pulls, he needs to buy the gacha pulls from the seller to continue pulling the gacha game, from which the seller can obtain revenue by selling the gacha pulls.

6.1.1 Subsidies in the Fixed-Probability Gacha Game. We consider the subsidies in the fixed-probability gacha game and show that subsidies degrade the seller's revenue. This is because subsidies in the fixed-probability gacha game can not motivate the buyer to buy more gacha pulls.

Theorem 6.1. *The subsidies in fixed-probability gacha game always degrade the seller's revenue compared to the gacha game without any subsidies.*

6.1.2 Subsidies in Varying-Probability Gacha Game with the Whale Property. We now consider the subsidies in the varying-probability gacha game with the whale property. The following theorem shows the buyer's behavior and seller's revenue with subsidies in the varying-probability gacha game with the whale property.

Theorem 6.2. *For a whale property gacha game, where the winning probability at state S_i is p_i and the cost for each gacha pull is c, if m free gacha pulls are subsidized in this gacha game, only the* *buyer with valuation greater than* $\varphi_s(m) = \left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=m}^{i-1}(1-p_j)\right) \cdot c$ *will buy the gacha pulls when they run out all the free gacha pulls, and the seller's revenue with m free gacha pulls is*

$$U_s(m) = \left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=1}^{i-1}(1-p_j)\right) \cdot c \cdot \left(1 - F\left(\left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=m+1}^{i-1}(1-p_j)\right) \cdot c\right)\right).$$

When $\arg\max U_s(m) > 0$*, subsidies can improve the seller's revenue.*

Consider a whale property gacha game with increasing probability, $\varphi_s(m)$ in Theorem 6.2 is monotonically decreasing with the number of free gacha pulls m. This implies that when the buyer's valuation is too low, subsidies can motivate the buyers to buy the gacha pulls, and thus improve the seller's revenue. Besides, compared to subsidies in the fixed-probability gacha game in Theorem 6.1, subsidies in the varying-probability gacha game make the sellers more controllable on the buyer's behavior, and can help to improve the seller's revenue. This is one of the factors contributing to the current widespread use of the varying-probability mechanism in the gacha games [50].

6.2 Subsidies and Grinding Behavior in Banner-based Multi-Item Gacha Game

In this section, we study the subsidies in the banner-based multi-item gacha game, where the subsidized free gacha pulls are available through all banners. Following the assumption in Section 5.2, we assume that the banner-based gacha game adopts the reset-after-winning mechanism and the buyer's state will be reset whenever he wins. We show that these subsidies will lead to the buyer's grinding behavior, that is, the buyer may accumulate the subsidized free gacha pulls, which is shown in the following theorem.

Theorem 6.3. *Consider a banner-based multi-item gacha game where there are K banners, and the buyer's valuations on the items in these banners are* $R_1, R_2, \ldots, R_K$*. Suppose the buyer is currently at the* i*-th banner with m free gacha pulls subsidized by the seller. There are three possible scenarios:*

- *If* $R_i \geq E(\pi_\infty) \cdot c$*, the buyer will use his free gacha pulls to pull the gacha game in this banner, and if he uses out all the free gacha pulls, he will buy the gacha pulls until he wins in this banner.*
- *If* $R_i < E(\pi_\infty) \cdot c$*, and there exists a banner* $j (i < j \leq K)$*, such that* $R_j \geq E(\pi_\infty) \cdot c$*, then the buyer would not buy any gacha pull before the* j*-th banner. In the* j*-th banner, he will first try to use all free gacha pulls, and then start to buy gacha pulls, until he finally wins the game.*
- *If* $R_j < E(\pi_\infty) \cdot c, \forall j \in [i, K]$*, let* $k^* = \arg\max_{k \in [i,K]} R_k$*, then the buyer would not buy any gacha pull before the* k^**-th banner, and will use the free gacha pulls to pull in the* k^**-th banner until he wins. If* $R_{k^*} < E(\pi_\infty, S_{m+1}) \cdot c$*, when the buyer uses out all the free gacha pulls in the* k^**-th banner, he will stop pulling the gacha. Otherwise,* $R_{k^*} \geq E(\pi_\infty, S_{m+1}) \cdot c$*, the buyer will buy the gacha pulls and pull in the* k^**-th banner until he wins.*

Here $E(\pi_\infty)$ *denotes the expected number of gacha pulls needed to win the gacha game, and* $E(\pi_\infty, S_{m+1})$ *denotes the expected number of gacha pulls needed to win the gacha game when the buyer is at state* S_{m+1}*, which are formulated in Lemma 3.3.*

Theorem 6.3 shows that in banner-based multi-item gacha games, subsidizing the buyer with free gacha pulls may not encourage the buyer to buy more gacha pulls. Instead, the buyer may accumulate the subsidized free gacha pulls and only buy some gacha pulls on some most valuable banners. This grinding behavior indeed harms the seller's revenue. Here we illustrate the buyer's grinding behavior in the following example.

Example 4. Consider a banner-based game with 2 banners. The buyer's valuations of the items in these 2 banners are $R_1 = 50$ and $R_2 = 100$. Each banner is a gacha game where $N = 100$, $p_i = 0.01, \forall i < N$ and $p_N = 1$. According to Theorem 5.2, without any subsidies, the buyer will only pull in the second banner, which will lead to the expected seller's revenue of 63.397. If the seller subsidizes the buyer as it does in the single-item gacha game, according to Theorem 6.2, the seller should give the buyer 32 free gacha pulls in the first banner and no free gacha pull in the second banner, assuming that these subsidies will encourage the buyer to pull in the first banner. However, Theorem 6.3 shows that a rational buyer will accumulate these subsidies and only buy the gacha pull in the second banner, resulting in the lower expected seller's revenue of 35.895. In this case, the subsidies lead to the buyer's grinding behavior and harm the seller's revenue.

Grinding behavior is prevalent in many free-to-play video games [45], which brings challenges for game subsidy design. To mitigate the negative impact of the grinding behavior on the seller's revenue, the seller should carefully design the order of the banner in the multi-item gacha game. As shown in the example above, swapping those two banners can mitigate the grinding problem. Besides, the seller can keep the incoming banner as a secret. Without knowing the details of the incoming banners, the buyer can not make a future plan, which can also mitigate the grinding problem. This is also a common tactic for video game firms. Figuring out the optimal game design with the subsidy scheme in the banner-based multi-item gacha game can be future work.

7 CASE STUDY: BLOCKCHAIN MINING AS A GACHA GAME

There are a wide range of potential applications of the gacha games in business management and social life, such as license plate lottery and probabilistic selling [30, 39, 46]. In this section, as a case study, we illustrate the interesting linkage between the gacha game model and the blockchain mining mechanisms. With such a correspondence established, the results derived in gacha game model (and possibly, results from a large body of literature in auction theory, as we have shown a simple relationship between gacha game and auction) could shed lights on the design of blockchain systems.

Blockchain is a distributed ledger with a sequence of blocks, which contain the transaction records [32]. The blocks in the blockchain are stochastically generated by the miners or the validators based on the consensus algorithms. The miners/validators should invest their computing power or stake their coins to win the right of generating a block with a certain probability. Once a block is generated, the corresponding miner/validator can obtain the block reward, which motivates the miners/validators to maintain the system consensus. Here, we investigate the blockchain systems with the two most common consensus algorithms: Proof-of-Work (PoW) [34] and Proof-of-Stake (PoS) [38] in detail and show that the blockchain system can be modeled as the gacha game, which is shown in Table 1. The miners in PoW blockchain or the validators in the PoS blockchain can be modeled as the buyers in the gacha game. And the blockchain system designer is modeled as the seller in the gacha game, who aims to maximize security assurance measured by the amount of the invested computing power in the PoW blockchain or invested coins in the PoS blockchain.

Gacha Game	Blockchain	
	PoW Blockchain	PoS Blockchain
buyer	miner	validator
seller	system designer	
gacha reward	block reward	
gacha pull	hash operation with nonce	hash operation with time
winning probability	probability that hash value hits the target	
price of each gacha pull	computing cost for hash operation	staking cost at time
seller's revenue	system's security guarantee	
	invested computing power	invested coins
optimal configuration	mining difficulty adjustment	
gacha game type	fixed-probability gacha game	gacha games in Table 2

Table 1. Blockchain as a gacha game

7.1 Proof-of-Work Blockchain as a Gacha Game

We first show that the blockchain system with Proof-of-Work (PoW) can be modeled as a fixed-probability gacha game. In the PoW blockchain, to generate a new block, a miner needs to find a nonce such that the hash value is less than D, where D is the pre-determined mining difficulty. The PoW blockchain can be modeled as a gacha game, where the miners act as the gacha game buyers; each hash operation is modeled as one gacha pull; the computing cost for each hash operation can be considered as the price of one gacha pull; when the miner finds a nonce such that the hash value is less than D, the miner can generate a new block and obtain the block reward R, which is equivalent to the case that the buyer wins the gacha game; and the probability of creating a valid block of each hash operation is $p = D/2^M$, where M is the number of bits for the hash value, and this probability p corresponds to the winning probability in the gacha game. If the blockchain designer wants to maximize the security guarantee in the blockchain system, where the security guarantee in the PoW blockchain can be measured by the amount of invested computing power, he needs to figure out the optimal design as shown in Theorem 4.10. Since the block reward and the cost of one hash operation change over time, the blockchain designer needs to adjust the probability p to achieve optimality. Generally, p is adjusted by the mining configuration D, which is exactly the mining difficulty adjustment in most of the PoW blockchains.

7.2 Proof-of-Stake Blockchain as a Gacha Game

Gacha Game	coins as stake	coin age as stake	
		linear coin age	Reddcoin (PoSV)
gacha game type	fixed-probability gacha game	gacha game with increasing probability	non-whale property gacha game
whale property	✓	✓	✗
reset-after-winning mechanism in sequential gacha game	✗	coin age resets to 0 when the validator finds a new block	
succeed-after-opt-out mechanism in banner-based gacha game	✗	coin age does not reset to 0 when others find a new block	

Table 2. PoS blockchain as a gacha game

In the PoS protocol, the validators of the blockchains create a valid block if a candidate block satisfies the condition that $\text{Hash}(\texttt{time}, ...) < D \cdot \texttt{stake}$ where `time` represents the timestamp when the candidate block is generate, D is the pre-determined mining difficulty and `stake` is the value of stakes possessed. The exact definition of `stake` varies from implementation to implementation.

The PoS blockchain can be modeled as a gacha game, where the validators act as the gacha game buyers; each hash operation at each time is modeled as one gacha pull; the staking cost per unit can be considered as the price of one gacha pull c; when the hash value of the candidate block is less then $D \cdot \texttt{stake}$ at a certain time, the validator can generate a new block and obtain the block reward R, which is equivalent to the case that the buyer wins the gacha game; and the probability of creating a valid block is $p = (D \cdot \texttt{stake})/2^M$, where M is the number of bits for the hash value, and this probability p can be modeled as the probability of winning the gacha game. The system designer of the PoS blockchain acts as the seller in the gacha game and wants to maximize the security guarantee, which can be measured by the amount of invested tokens in the PoS blockchain. We further illustrate such correspondence by the following typical PoS blockchains.

7.2.1 PoS Blockchains with Tokens as Stake. For the PoS blockchains that define the stake as the number of tokens, such as Blackcoin [43], the probability that a validator creates a block at each time is the same. These PoS blockchains can be modeled as the fixed-probability gacha games.

7.2.2 PoS Blockchains with "Coin Age" as Stake. Some blockchains such as Peercoin [25] define a validator's state as "coin age", which is the product of the number of tokens and the amount of time that a single validator has held them. Therefore, the probability that a validator creates a valid block is increasing monotonically. Specifically, the probability of winning the gacha (creating a valid block) at the i-th gacha pull (hash operation at time i) is $p_i = \min\left\{\frac{D \cdot \#tokens \cdot i}{2^M}, 1\right\}$, where $\#tokens$ denotes the number of tokens owned by the validator. We can model these PoS blockchains with "coin age" as stake as the gacha games with increasing probability, where the coin age works like the pity counter in the pity system [50]. The increasing probability in the blockchain can resolve the problem that some unlucky validators may not produce blocks in a long time and reduce the variance of validators' income.

7.2.3 Multi-item Gacha Game in PoS Blockchain. The generation process of each block can be modeled as a single-item gacha game, and the whole block chain generation process can be modeled as a multi-item gacha game. In most of the PoS blockchains with "coin age", the coin age of the validator will be accumulated until he successfully generates a new block. This is similar to the succeed-after-opt-out mechanism in the banner-based gacha game in Section 5.2, where the behavior that the buyer chooses to opt-out in the banner-based gacha game corresponds to the scenario that other validators find a new block in the PoS blockchain, and the buyer's state will be carried to the next banner. Besides, once the validator generates a new block, his accumulated coin age will be reset to 0, which is similar to the reset-after-winning mechanism in the sequential gacha game in Section 5.1.

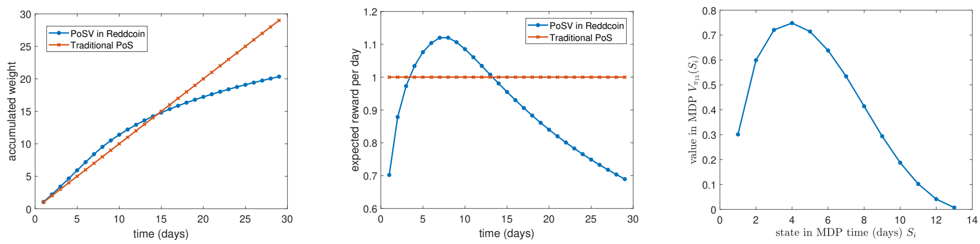
7.2.4 PoS Blockchain as a non-Whale Property Gacha Game. Reddcoin is a special PoS blockchain, whose consensus protocol is Proof of Stake Velocity (PoSV) [36], a variation of the traditional PoS. According to the source code of Reddcoin (line 66-85 in file "reddcoin/src/kernal.cpp") [3], the coin age weight in Reddcoin is calculated as follows:

$$\text{weight} = \begin{cases} -0.00408163 * \text{time}^3 + 0.05714286 * \text{time}^2 + \text{time}, & \text{time} \leq 7\text{days}, \\ 8.4 * \log(\text{time}) - 7.94564525, & \text{otherwise}. \end{cases} \tag{3}$$

Figure 3 (a) shows the accumulated weight of the coin age in Reddcoin, which is calculated based on equation (3). We can find that with the non-linear accumulation of coin age, the coin age in Reddcoin is accumulated at a higher rate than in traditional PoS initially, and the marginal accumulation decreases over time. Figure 3 (b) shows the expected reward per day in Reddcoin and traditional PoS. By modeling the Reddcoin as a gacha game, with each day staking as a gacha pull, the expected reward per day in Reddcoin is consistent with $p_i R$ in the gacha game model. Note that in the traditional PoS with coin age, the normalized expected reward per day is 1 [2], which can be considered as the cost $c = 1$ in the gacha game model. With the expected reward per day $p_i R$ and the cost c, we can apply the gacha game model to analyze the Reddcoin system. According to Lemma 3.1 and Lemma 3.2, the optimal policy for the validator in Reddcoin (the buyer in the gacha game) is π_{13}, which implies that Reddcoin is a non-whale property gacha game. Figure 3 (c) shows the value $V_{\pi_{13}}(S_i)$ in MDP of the gacha game model of Reddcoin. The rational validator should stake to the system initially, and then update his staking after 14 days. This theoretical analysis is highly consistent with the staking phenomenon in Reddcoin, where staking regularly for 2 weeks (14 days) can achieve the highest validator's revenue [2]. Recall that the non-whale property gacha game can not achieve the highest seller's revenue but can encourage more people to join the game. This is also the core idea of Reddcoin. Reddcoin encourages users to use the coins instead of staking them, which can promote a better coin economy by increasing the participation of coin holders [36].

8 CONCLUSION

In this work, we propose a mathematical model for the gacha game, where the sequential decision of the buyer is modeled as a Markov Decision Process (MDP). We introduce the definition of whale



(a) Accumulated weight of the coin age in Reddcoin and traditional PoS (b) Expected reward per day in Reddcoin and traditional PoS (c) Value $V_{\pi_{13}}(S_i)$ in MDP of the gacha game model of Reddcoin

Fig. 3. Gacha game model of Reddcoin

property and further show the optimality of the whale property gacha game and the equivalence of the gacha game and the single-item single-bidder Myerson auction. We also provide an optimal gacha game configuration that can achieve the maximum seller's revenue. Besides, we further explore the multi-item gacha games, including the sequential multi-item gacha game and the banner-based multi-item gacha game. Moreover, we discuss the subsidies in the gacha game and model the buyer's grinding behavior with subsidies. Finally, we offer a case study of the gacha game on the blockchain system.

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A PROOFS IN SECTION 3

Lemma 3.1 *The value of MDP of the policy π_k at state S_i is*

$$\begin{aligned}V_{\pi_k}(S_i) &= \left(p_i + \sum_{m=i+1}^{k} p_m \prod_{j=i}^{m-1}(1-p_j)\right) R - \left(1 + \sum_{m=i+1}^{k}\prod_{j=i}^{m-1}(1-p_j)\right) c, \quad \forall i \le k \\ &= \left(1 - \prod_{j=i}^{k}(1-p_j)\right) R - \left(p_i + \sum_{m=i+1}^{k}(m-i+1)p_m \prod_{j=i}^{m-1}(1-p_j) + (k-i+1)\prod_{j=i}^{k}(1-p_j)\right) c\end{aligned}$$

Proof. According to the MDP, we have

$$V_{\pi_k}(S_k) = p_k R - c,$$

and

$$V_{\pi_k}(S_i) = p_i R - c + (1-p_i)V_{\pi_k}(S_{i+1}), i < k,$$

Recursively, we have

$$V_{\pi_k}(S_{k-i}) = p_{k-i}R - c + \sum_{m=0}^{i-1}(p_{k-m}R - c)\prod_{j=k-i}^{k-m-1}(1-p_j).$$

Therefore we have

$$\begin{aligned}V_{\pi_k}(S_i) &= p_i R - c + \sum_{m=0}^{k-i-1}(p_{k-m}R - c)\prod_{j=i}^{k-m-1}(1-p_j) \\ &= p_i R - c + \sum_{m=i+1}^{k}(p_m R - c)\prod_{j=i}^{m-1}(1-p_j) \\ &= \left(p_i + \sum_{m=i+1}^{k} p_m \prod_{j=i}^{m-1}(1-p_j)\right) R - \left(1 + \sum_{m=i+1}^{k}\prod_{j=i}^{m-1}(1-p_j)\right) c, \quad \forall i \le k.\end{aligned}$$

To prove the lemma, we only need to prove the following two mathematical claims:
Claim 1: $\left(p_i + \sum_{m=i+1}^{k} p_m \prod_{j=i}^{m-1}(1-p_j)\right) = \left(1 - \prod_{j=i}^{k}(1-p_j)\right)$.
Proof 1: For convenience, we denote that $\bar{p_i} = 1 - p_i$. Then we have

$$\begin{aligned}&\left(p_i + \sum_{m=j+1}^{k} p_m \prod_{j=i}^{m-1}(1-p_j)\right) + (1-p_k)\prod_{j=i}^{k-1}(1-p_j) \\ =& p_i + p_{i+1}\bar{p_i} + p_{i+2}\bar{p_i}\bar{p_{i+1}} + \cdots + p_k\bar{p_i}\bar{p_{i+1}}\cdots\bar{p_{k-1}} + \bar{p_k}\bar{p_i}\bar{p_{i+1}}\cdots\bar{p_{k-1}} \\ =& p_i + p_{i+1}\bar{p_i} + p_{i+2}\bar{p_i}\bar{p_{i+1}} + \cdots + \bar{p_i}\bar{p_{i+1}}\cdots\bar{p_{k-1}} \\ =& p_i + \bar{p_i}\left(p_{i+1} + \bar{p_{i+1}}\left(p_{i+2} + \bar{p_{i+2}}\left(\cdots + \bar{p_{k-2}}\left(p_{k-1} + \bar{p_{k-1}}\right)\right)\right)\right) \\ =& p_i + \bar{p_i}\left(p_{i+1} + \bar{p_{i+1}}\left(p_{i+2} + \bar{p_{i+2}}\left(\cdots + \left(p_{k-1} + \bar{p_{k-1}}\right)\right)\right)\right) \\ =& p_i + \bar{p_i} = 1\end{aligned} \tag{4}$$

Then Claim 1 can be prove by moving $\prod_{j=i}^{k}(1-p_j)$ to the right side of the equation (4).
Claim 2:

$$\left(1 + \sum_{m=i+1}^{k}\prod_{j=i}^{m-1}(1-p_j)\right) = \left(p_i + \sum_{m=i+1}^{k}(m-i+1)p_m\prod_{j=i}^{m-1}(1-p_j) + (k-i+1)\prod_{j=i}^{k}(1-p_j)\right)$$

Proof 2: We first prove that

$$p_i + \sum_{m=i+1}^{k} ((m-i+1)p_m - 1) \prod_{j=i}^{m-1} (1-p_j) + (k-i+1) \prod_{j=i}^{k} (1-p_j) = 1$$

The proof goes as follows.

$$\begin{aligned}
&p_i + \sum_{m=i+1}^{k} ((m-i+1)p_m - 1) \prod_{j=i}^{m-1} (1-p_j) \\
=&p_i + (2p_{i+1} - 1)\bar{p_i} + (3p_{i+2} - 1)\bar{p_i}\bar{p_{i+1}} + \cdots + ((k-i)p_{k-1} - 1)\,\bar{p_i}\bar{p_{i+1}} \cdots \bar{p_{k-2}} \\
&\quad + ((k-i+1)p_k - 1)\bar{p_k}\bar{p_k} \cdots \bar{p_{k-1}} + (k-i+1)\bar{p_k}\bar{p_k} \cdots \bar{p_{k-1}} \\
=&p_i + (2p_{i+1} - 1)\bar{p_i} + (3p_{i+2} - 1)\bar{p_i}\bar{p_{i+1}} + \cdots + ((k-i)p_{k-1} - 1)\,\bar{p_i}\bar{p_{i+1}} \cdots \bar{p_{k-2}} + (k-i)\bar{p_k}\bar{p_k} \cdots \bar{p_{k-1}} \\
=&p_i + \bar{p_i}\left((2p_{i+1} - 1) + \bar{p_{i+1}}\left((3p_{i+2} - 1) + \bar{p_3}\left(\cdots \bar{p_{k-2}}\left((k-i)p_{k-1} + (k-i)\bar{p_{k-1}}\right)\right)\right)\right) \\
=&p_i + \bar{p_i}\left((2p_{i+1} - 1) + \bar{p_{i+1}}\left((3p_{i+2} - 1) + \bar{p_{i+2}}\left(\cdots (k-2)p_{k-2} - 1 + \bar{p_{k-2}}\,(k-2)\right)\right)\right) \\
=&p_i + \bar{p_i}\left((2p_{i+1} - 1) + 2\bar{p_{i+1}}\right) \\
=&p_i + \bar{p_i} = 1
\end{aligned} \tag{5}$$

By adding $\sum_{m=i+1}^{k} \prod_{j=i}^{m-1} (1-p_j)$ to both sides of the equation (5), Claim 2 can be proved. The proof is thus completed. □

Lemma 3.2 *The policy π_k at state S_i is optimal if and only if the following conditions are satisfied:*

(1) $V_{\pi_k}(S_j) \geq 0, \quad \forall j \in [i, k]$.
(2) $V_{\pi_l}(S_{k+1}) \leq 0, \quad \forall l \in [k+1, \infty)$.

Proof. We first show that if Condition (1) and (2) are satisfied, the policy π_k is the optimal policy. Here we adopt the backward induction (value iteration) to obtain the optimal policy. As mentioned in Section 3.1, there are two possible actions at each state, namely, a_p and a_q. The action a_p means that the buyer will pull the gacha game, and the action a_q means that the buyer will quit. According to Condition (2), we have that

$$\max_{l \in [k+1, \infty)} V_{\pi_l}(S_{k+1}) \leq 0.$$

Therefore, we have

$$\begin{aligned}
V(S_{k+1}) &= \max_{a \in \{a_p, a_q\}} \left(\sum_{S'} P_a(S_{k+1}, S')(R_a(S_{k+1}, S') + V(S')) \right) \\
&= \max \left(0, \max_{l \in [k+1, \infty)} V_{\pi_l}(S_{k+1}) \right) = 0.
\end{aligned}$$

Thus, the optimal action at state S_{k+1} is

$$\pi(S_{k+1}) = \arg\max_{a \in \{a_p, a_q\}} \left(\sum_{S'} P_a(S_{k+1}, S')(R_a(S_{k+1}, S') + V(S')) \right) = a_q.$$

According to Condition (1), we have that

$$\max_{l \in [j, k]} V_{\pi_l}(S_j) \geq V_{\pi_k}(S_j) \geq 0, \ \forall j \in [i, k].$$

Therefore, we have

$$\begin{aligned}V(S_j) &= \max_{a\in\{a_p,a_q\}}\left(\sum_{S'} P_a(S_j,S')(R_a(S_j,S')+V(S'))\right)\\ &= \max\left(0, \max_{l\in[j,\infty)} V_{\pi_l}(S_j)\right)\\ &= \max\left(0, \max\left(\max_{l\in[j,k]} V_{\pi_l}(S_j), \max_{l\in[k+1,\infty)} V_{\pi_l}(S_j)\right)\right)\\ &\geq \max_{l\in[j,k]} V_{\pi_l}(S_j) \geq 0, \quad \forall j\in[i,k].\end{aligned}$$

Thus, the optimal action at state S_j, $\forall j \in [i,k]$ is

$$\pi(S_j) = \arg\max_{a\in\{a_p,a_q\}}\left(\sum_{S'} P_a(S,S')(R_a(S,S')+V(S'))\right) = a_p.$$

Therefore, we have that

$$\pi(S_j) = a_p,\ \forall j\in[i,k],\ \text{and}\ \pi(S_{k+1}) = a_q,$$

which means that the buyer should continue pulling the gacha game until he wins or reaches state S_{k+1}. Thus, when Condition (1) and (2) are satisfied, the optimal policy for the buyer at state S_i is π_k.

We next show that if the policy π_k at state S_i is optimal, then Condition (1) and (2) are satisfied.

Assume that Condition (1) does not hold, i.e., $\exists \hat{j} \in [i,k]$, s.t., $V_{\pi_k}(S_{\hat{j}}) < 0$. Suppose that $\exists t \in [i+1,\hat{j}]$, s.t. $V_{\pi_{\hat{j}-1}}(S_t) > V_{\pi_k}(S_t)$, then we have

$$\begin{aligned}&V_{\pi_{\hat{j}-1}}(S_{t-1}) - V_{\pi_k}(S_{t-1})\\ =&\left((p_tR-c)+(1-p_{t-1})V_{\pi_{\hat{j}-1}}(S_t)\right) - \left((p_tR-c)+(1-p_{t-1})V_{\pi_k}(S_t)\right)\\ =&(1-p_{t-1})\left(V_{\pi_{\hat{j}-1}}(S_t) - V_{\pi_k}(S_t)\right) > 0,\end{aligned}$$

which means

$$V_{\pi_{\hat{j}-1}}(S_t) > V_{\pi_k}(S_t) \Rightarrow V_{\pi_{\hat{j}-1}}(S_{t-1}) > V_{\pi_k}(S_{t-1}), \quad t\in[i+1,\hat{j}].$$

As we know that $V_{\pi_{\hat{j}-1}}(S_{\hat{j}}) = 0$, and thus $V_{\pi_{\hat{j}-1}}(S_{\hat{j}}) > V_{\pi_k}(S_{\hat{j}})$. By mathematical induction, we have that

$$V_{\pi_{\hat{j}-1}}(S_t) > V_{\pi_k}(S_t), \quad \forall t\in[i,\hat{j}].$$

Specially, we have $V_{\pi_{\hat{j}-1}}(S_1) > V_{\pi_k}(S_1)$, which implies that the policy $\pi_{\hat{j}-1}$ is better than the policy π_k, which leads to contradiction.

Assume that Condition (2) does not hold, i.e., $\exists \hat{l} \in [k+1,\infty)$, s.t., $V_{\pi_{\hat{l}}}(S_k) > 0$. Suppose that $\exists t \in [i+1,k+1]$, s.t. $V_{\pi_{\hat{l}}}(S_t) > V_{\pi_k}(S_t)$, then we have

$$\begin{aligned}&V_{\pi_{\hat{l}}}(S_{t-1}) - V_{\pi_k}(S_{t-1})\\ =&\left((p_tR-c)+(1-p_{t-1})V_{\pi_{\hat{l}}}(S_t)\right) - \left((p_tR-c)+(1-p_{t-1})V_{\pi_k}(S_t)\right)\\ =&(1-p_{t-1})\left(V_{\pi_{\hat{l}}}(S_t) - V_{\pi_k}(S_t)\right) > 0,\end{aligned}$$

which means

$$V_{\pi_{\hat{l}}}(S_t) > V_{\pi_k}(S_t) \Rightarrow V_{\pi_{\hat{l}}}(S_{t-1}) > V_{\pi_k}(S_{t-1}), \quad t\in[i+1,k+1].$$

As we know that $V_{\pi_k}(S_{k+1}) = 0$, and thus $V_{\pi_{\hat{l}}}(S_{k+1}) > V_{\pi_k}(S_{k+1})$. By mathematical induction, we have that

$$V_{\pi_{\hat{l}}}(S_t) > V_{\pi_k}(S_t), \quad \forall t \in [i, k+1].$$

Specially, we have $V_{\pi_{\hat{l}}}(S_1) > V_{\pi_k}(S_1)$, which implies that the policy $\pi_{\hat{l}}$ is better than the policy π_k, which leads to contradiction. This proof is thus completed. □

Lemma 3.3 *For the buyer who adopts policy π_k and is currently at state S_i, the expected number of gacha pulls that the buyer will buy is*

$$E(\pi_k, S_i) = \begin{cases} 0, & k < i, \\ p_i + \sum_{j=i+1}^{k}(j-i+1)p_j \prod_{l=i}^{j-1}(1-p_l) + (k-i+1)\prod_{j=i}^{k}(1-p_j), & k \geq i. \end{cases}$$

Proof. The probability of winning the gacha game at state S_j ($j \geq i$) is

$$q(j) = \begin{cases} p_i, & j = i, \\ p_j \prod_{l=i}^{j-1}(1-p_l), & j > i. \end{cases}$$

Therefore, the expected number of gacha pulls needed to win the gacha game for a buyer with policy π_k at state S_i is

$$\begin{aligned} E(\pi_k, S_i) &= (k-i+1)\prod_{j=i}^{k}(1-p_j) + \sum_{j=i}^{k} q(j)\cdot(j-i+1) \\ &= p_i + \sum_{j=i+1}^{k}(j-i+1)p_j\prod_{l=i}^{j-1}(1-p_l) + (k-i+1)\prod_{j=i}^{k}(1-p_j), \quad \forall k \geq i. \end{aligned}$$

When $k < i$, the buyer will never buy any gacha pull. Therefore, the expected number of gacha pulls that the buyer will buy is 0. The proof is thus completed. □

B PROOFS IN SECTION 4

Lemma 4.2 *The optimal policy π^* for the buyer in the gacha game with the whale property is*

$$\pi^* = \begin{cases} \pi_\infty, & \textit{When } V_{\pi_\infty}(S_1) \geq 0 \Leftrightarrow R \geq E(\pi_\infty)c, \\ \pi_0, & \textit{When } V_{\pi_\infty}(S_1) < 0 \Leftrightarrow R < E(\pi_\infty)c. \end{cases}$$

Proof. According to Lemma 3.1, the value in MDP at the initial state S_1 for a buyer with valuation R and policy π_k is

$$V_{\pi_\infty}(S_1) = R - \left(p_1 + \sum_{m=2}^{\infty} m p_m \prod_{i=1}^{m-1}(1-p_i)\right)c. \tag{6}$$

Besides, according to Lemma 3.3, we have that

$$E(\pi_\infty) = \left(p_1 + \sum_{m=2}^{\infty} m p_m \prod_{i=1}^{m-1}(1-p_i)\right).$$

Therefore, Equation (6) can be rewritten as

$$V_{\pi_\infty}(S_1) = R - E(\pi_\infty)c.$$

Thus, the optimal policy π^* in the gacha game with the whale property is straightforward, which is listed as follows:

$$\begin{cases} \text{When } V_{\pi_\infty}(S_1) \geq 0 \Leftrightarrow R \geq E(\pi_\infty)c, & \pi^* = \pi_\infty, \\ \text{When } V_{\pi_\infty}(S_1) < 0 \Leftrightarrow R < E(\pi_\infty)c, & \pi^* = \pi_0. \end{cases}$$

The proof is thus completed. □

Lemma 4.3 *If the probability of winning the gacha game is monotonically increasing for each gacha pull, i.e., $p_i \leq p_{i+1}, \forall i$, the gacha game has the whale property.*

Proof. We can prove this lemma by contradiction. If the gacha game is not whale, there exists a buyer with valuation R whose optimal policy is $\pi_k, k \neq 0$ such that $V_{\pi_k}(S_j) > V_{\pi_i}(S_j), \forall i > k, j \leq k$, which means that the buyer will stop pulling the gacha after k rounds. Then the value at state S_k in the MDP is

$$V_{\pi_k}(S_k) = p_k R - c.$$

which should greater or equal to zero, otherwise the buyer should stop earlier and π_k will not be the optimal policy. Since $p_{k+1} \geq p_k$, we have that

$$V_{\pi_i}(S_{k+1}) = p_{k+1}R - c + (1 - p_{k+1})V_{\pi_i}(S_{k+2}) \geq p_{k+1}R - c \geq V_{\pi_k}(S_k), \quad i > k,$$

which leads to contradiction. Therefore, if $p_i \leq p_{i+1}, \forall i$, the gacha game has the whale property. The proof is thus completed. □

Lemma 4.5 *When the buyer's valuation follows the discrete distribution $P(R = R_i) = \beta_i, \quad i = 1, 2, \cdots, M$, the whale property gacha game with equivalent probability $\tilde{p} = \hat{p}_{i^*}$ can achieve the maximum seller's revenue $\frac{c}{\hat{p}_{i^*}} \sum_{j=i^*}^{M} \beta_j$, where*

$$i^* = \arg \max_{i \in \{1,2,\cdots,M\}} \frac{c}{\hat{p}_i} \sum_{j=i}^{M} \beta_j, \quad \textit{and} \quad \hat{p}_i = \frac{c}{R_i} \leq 1.$$

Proof. The whale property of the gacha game implies that the buyer will either continue pulling the gacha or just leave it. Suppose that in a whale property gacha game, the buyer with valuation greater than R_i will pull the gacha and those with smaller valuation will just leave. Then we know that the equivalent probability should be $\frac{c}{R_i} \leq \tilde{p} < \frac{c}{R_{i-1}}$. Besides, fixing the price of each gacha pull c, the smaller $\tilde{p}$ implies a large $E(\pi_\infty)$ and possibly a higher revenue. Therefore, in that scenario, the optimal equivalent probability is $\tilde{p} = \hat{p}_i = \frac{c}{R_i}$, and can achieve the revenue $\frac{c}{\hat{p}_i} \sum_{j=i}^{M} \beta_j$. By enumerating $i = 1, 2, \cdots, M$, we can find the optimal configuration. The proof is thus completed. □

Lemma 4.7 *When the buyer's valuation follows the discrete distribution $P(R = R_i) = \beta_i, \quad i = 1, 2, \cdots, M$. Consider a gacha game $\mathbb{G}^*$ that can be divided $L (L \leq M)$ whale property subgames, and the i-th whale property subgame contains n_i rounds of gacha pulls, i.e., $n_i = a_i - a_{i-1}, n_i \geq 0$, where $i = 1, 2, \cdots, M$. Specially, the length of i-th subgame being $n_i = 0$ implies that the i-th whale property subgame is dummy. The equivalent probability of the i-th subgame is $\tilde{p}_i = c/R_i (c \leq R_i)$ and the lengths of these whale property subgame $\boldsymbol{n} = (n_1, n_2, \cdots, n_M)$ are*

$$\arg\max_{\boldsymbol{n}} c \cdot \sum_{k=1}^{M} \beta_k Q_k(\boldsymbol{n}), \quad n_i \geq 0, \forall i = 1, 2, \cdots, M.$$

where $Q_k(\boldsymbol{n})$ is the expected number of gacha pulls bought by the buyer with valuation R_k in the gacha game with the lengths of the whale property subgames $\boldsymbol{n}$, which is formulated as follows:

$$Q_k(\boldsymbol{n}) = \begin{cases} \frac{1-(1-\tilde{p}_1)^{n_1}}{\tilde{p}_1}, & k = 1, \\ \left(\sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} (1 - \tilde{p}_j)^{n_j} \right) \frac{1-(1-\tilde{p}_i)_i^n}{\tilde{p}_i} \right), & k > 1. \end{cases}$$

Then the gacha game $\mathbb{G}^*$ is optimal and can achieve the maximum seller's revenue.

Proof. If the optimal gacha game is the whale property gacha game with equivalent probability p_k, then it corresponds to the scenario where $n_i = 0, \forall i \neq k$. So we only need to consider the non-whale property gacha game. Without the whale property, we assume that the buyer with valuation R_k will continue pulling the gachas and stop at state $S_{\sum_{i=1}^{k} n_i}$. Then we know that the subgame $\mathbb{G}(\sum_{i=1}^{k-1} n_i, \sum_{i=1}^{k} n_i)$ has the whale property because none of the buyers will stop at the midway between state $S_{\sum_{i=1}^{k-1} n_i}$ and state $S_{\sum_{i=1}^{k} n_i}$. Since the buyer with valuation greater or equal to R_k will pull in the subgame $\mathbb{G}(\sum_{i=1}^{k-1} n_i, \sum_{i=1}^{k} n_i)$, we know that the equivalent probability of the subgame $\mathbb{G}(\sum_{i=1}^{k-1} n_i, \sum_{i=1}^{k} n_i)$ should be at least c/R_k. To achieve the maximum revenue, the equivalent probability of the subgame $\mathbb{G}(\sum_{i=1}^{k-1} n_i, \sum_{i=1}^{k} n_i)$ should be c/R_k . By enumerating all the possible combinations of $n_i, i = 1, 2, \cdots, M$, we can find the optimal design for the general gacha game.

We next will show that the expected number of gacha pulls bought by the buyer with valuation R_k in the gacha game with configuration $\mathbf{n} = (n_1, n_2, \cdots, n_M)$ is

$$Q_k(\mathbf{n}) = \begin{cases} \frac{1-(1-\tilde{p}_1)^{n_1}}{\tilde{p}_1}, & k = 1, \\ \left(\sum_{i=1}^{k} \left(\prod_{j=1}^{i-1}(1-\tilde{p}_j)^{n_j}\right) \frac{1-(1-\tilde{p}_i)^{n}_{i}}{\tilde{p}_i}\right), & k > 1. \end{cases}$$

When $k = 1$, we have that

$$Q_1 = \sum_{i=1}^{n_1} \tilde{p}_1(1-\tilde{p}_1)^{i-1} \cdot i + (1-\tilde{p}_1)^{n_1} \cdot n_1 = \frac{1-(1-\tilde{p}_1)^{n_1}}{\tilde{p}_1}.$$

To prove the lemma above, we only need to prove that

$$Q_{k+1} - Q_k = \left(\prod_{i=1}^{k}(1-\tilde{p}_k)^{n_k}\right) \frac{1-(1-\tilde{p}_{k+1})^{n_{k+1}}}{\tilde{p}_{k+1}}.$$

We know that $\forall k > 1$, we have

$$Q_k = \left(\sum_{i=1}^{n_1} \tilde{p}_1(1-\tilde{p}_1)^{i-1} \cdot i + \sum_{i=2}^{k} \sum_{j=1+\sum_{t=1}^{i-1} n_t}^{\sum_{t=1}^{i} n_t} \left(\prod_{s=1}^{i-1}(1-\tilde{p}_s)^{n_s}\right) \cdot \tilde{p}_i\,(1-\tilde{p}_i)^{j-\sum_{t=1}^{i-1} n_t} \cdot i\right) + \left(\prod_{i=1}^{k}(1-\tilde{p}_i)^{n_i}\right) \cdot \left(\sum_{i=1}^{k} n_i\right).$$

Then we have that

$$\begin{aligned}
&Q_{k+1} - Q_k \\
&= \sum_{j=1+\sum_{t=1}^{k} n_t}^{\sum_{t=1}^{k+1} n_t} \left(\prod_{s=1}^{k}(1-\tilde{p}_s)^{n_s}\right) \cdot \tilde{p}_{k+1}\,(1-\tilde{p}_{k+1})^{j-\sum_{t=1}^{k} n_t} \cdot i + \left(\prod_{i=1}^{k+1}(1-\tilde{p}_i)^{n_i}\right) \cdot \left(\sum_{i=1}^{k+1} n_i\right) - \left(\prod_{i=1}^{k}(1-\tilde{p}_i)^{n_i}\right) \cdot \left(\sum_{i=1}^{k} n_i\right) \\
&= \left(\prod_{i=1}^{k}(1-\tilde{p}_k)^{n_k}\right)\left(\sum_{i=1}^{n_{k+1}} \tilde{p}_{k+1}(1-\tilde{p}_{k+1})^{i-1} \cdot \left(i + \sum_{j=1}^{k} n_k\right)\right) + \left(\prod_{i=1}^{k+1}(1-\tilde{p}_i)^{n_i}\right) \cdot \left(\sum_{i=1}^{k+1} n_i\right) - \left(\prod_{i=1}^{k}(1-\tilde{p}_i)^{n_i}\right) \cdot \left(\sum_{i=1}^{k} n_i\right) \\
&= \left(\prod_{i=1}^{k}(1-\tilde{p}_k)^{n_k}\right)\left(\sum_{i=1}^{n_{k+1}} \tilde{p}_{k+1}(1-\tilde{p}_{k+1})^{i-1} \cdot i + (1-\tilde{p}_{k+1})^{n_{k+1}} n_{k+1} + \sum_{j=1}^{k} n_k \left((1-\tilde{p}_{k+1})^{n_{k+1}} + \sum_{i=1}^{n_{k+1}} \tilde{p}_{k+1}(1-\tilde{p}_{k+1})^{i-1}\right) - \sum_{j=1}^{k} n_k\right) \\
&= \left(\prod_{i=1}^{k}(1-\tilde{p}_k)^{n_k}\right)\left(\sum_{i=1}^{n_{k+1}} \tilde{p}_{k+1}(1-\tilde{p}_{k+1})^{i-1} \cdot i + (1-\tilde{p}_{k+1})^{n_{k+1}} n_{k+1}\right) = \left(\prod_{i=1}^{k}(1-\tilde{p}_k)^{n_k}\right) \frac{1-(1-\tilde{p}_{k+1})^{n_{k+1}}}{\tilde{p}_{k+1}}
\end{aligned}$$

The proof is thus completed. □

Theorem 1 *The maximum seller's revenue of the non-whale property gacha game is less than that of the whale property gacha game.*

Proof. According to Lemma 4.7, we have that

$$c \cdot Q_k(\mathbf{n}) = \sum_{i=1}^{k} A_i,$$

where

$$A_k = \begin{cases} (1-\alpha_1)R_1, & k = 1, \\ \left(\prod_{i=1}^{k-1} \alpha_i\right)(1-\alpha_k)R_k, & k > 1, \end{cases}$$
$$\alpha_k = (1-\tilde{p}_k)^{n_k} \in [0,1].$$

According to Proposition 4.7, the optimal configuration to achieve the maximum seller's revenue can be found by solving the following optimization problem.

$$\max f(\alpha_1, \alpha_2, \cdots, \alpha_K) = (1-\alpha_1)R_1 + \sum_{k=2}^{K}\left(1 - \sum_{j=1}^{k-1}\beta_j\right)\left(\prod_{i=1}^{k-1}\alpha_i\right)(1-\alpha_k)R_k, \quad \alpha_i \in [0,1], \forall i \tag{7}$$

Let $\alpha_1^*, \cdots, \alpha_K^* = \arg\max f(\alpha_1, \alpha_2, \cdots, \alpha_K)$ and $k^* = \min\{i | \alpha_i^* = 0\}$. If $\alpha_i^* = 1, \forall 1 \leq i < k^*$, the optimal game configuration is a whale property gacha game, which is consistent with Proposition 4.5. Otherwise, if $\exists 1 \leq i < k^*, s.t., 0 < \alpha_i^* < 1$, the non-whale property gacha game can achieve the maximum revenue. However, if we calculate the partial derivative of the function $f(\alpha_1, \alpha_2, \cdots, \alpha_K)$, we have $\frac{\partial f}{\partial \alpha_i} = c_i$, where c_i is a constant, which implies that to achieve the maximum, α_i should be either 0 or 1. Therefore, only the whale property gacha game can achieve the maximum revenue.

We now have proved the optimality of the whale property gacha game when the buyer's valuation follows a discrete distribution. We further show that the results of the optimality of the whale property gacha game can be extended to the scenario where the buyers' valuation follows the general distribution F (continuous/discrete distribution and even the mixed continuous and discrete distribution), where $F(\cdot)$ is the cumulative distribution function.

Suppose that a gacha game with K whale property subgames can achieve the maximum seller's revenue when the buyer's valuation follows the distribution F. Denote R_i as the minimum valuation of the buyers who will pull in the i-th whale property subgame. Then $\beta_i = F(R_{i+1}) - F(R_i)$ denotes the proportion of buyers who will pull the i-th whale property subgame but will not pull the $(i+1)$-th subgame. Specially, $\beta_K = F(R_{K+1}) - F(R_K) = 1 - F(R_K)$ denotes the proportion of buyers who will pull all of the K subgames. $\beta_0 = F(R_1) - F(R_0) = F(R_1)$ denotes the proportion of buyers who will never pull the gacha game. Then we have that $\sum_{i=0}^{K} \beta_i = 1$. According to Lemma 4.7, we know that the equivalent probability of the i-th subgame is $p_i = c/R_i, i = 1, 2, \cdots, K$. Combining with Lemma 4.7, we can find the optimal gacha game design that can achieve the maximum seller's revenue by converting into an optimization problem shown in Equation (7). Similar to the proof above, we can show that only the whale property gacha game can achieve the maximum revenue even when the buyer's valuation follows a general distribution. The proof is thus completed. □

Lemma B.1. *Define opt(R) $\in \{0, 1, 2, \cdots\}$ as a function of the buyer's valuation R, where the optimal gacha pulling policy for a buyer with valuation R at the initial state S_1 is $\pi_{opt(R)}$. Then the function opt(R) is nondecreasing on R.*

Proof. Consider there are two buyers with their valuations R_1, R_2, we want to prove that if $R_1 \leq R_2$, then we have $\text{opt}(R_1) \leq \text{opt}(R_2)$. The proof goes as follows.

For convenience, we denote $V_{\pi_k}(S_i, R)$ as the value of MDP of the policy π_k at state S_i for a buyer with valuation R, which is consistent with the definition in Lemma 3.1. According to Lemma 3.1, we know that $V_{\pi_k}(S_i, R)$ is monotonically increasing on R, which implies that

$$V_{\pi_k}(S_i, R_1) \le V_{\pi_k}(S_i, R_2), \quad \forall k, i. \tag{8}$$

According to Lemma 3.2, since the optimal policy for the buyer with valuation R_1 at the initial state S_1 is $\pi_{\text{opt}(R_1)}$, we have that

$$V_{\pi_{\text{opt}(R_1)}}(S_j, R_1) \ge 0, \quad \forall j \in [1, \text{opt}(R_1)]. \tag{9}$$

Combining inequality of (8) and (9), we have that

$$V_{\pi_{\text{opt}(R_1)}}(S_j, R_2) \ge V_{\pi_{\text{opt}(R_1)}}(S_j, R_1) \ge 0, \quad \forall j \in [1, \text{opt}(R_1)],$$

which implies that $\text{opt}(R_2) \ge \text{opt}(R_1)$, otherwise, it leads to contradiction to the condition (1) in Lemma 3.2. The proof is thus completed. □

Theorem 4.9 *Consider a gacha game $\mathbb{G}$, which can be divided into L consecutive whale property subgames, namely, $\mathbb{G}(a_{i-1}+1, a_i), i = 1, 2, \cdots, L$, where $a_0 = 0$ and $a_L = \infty$. The gacha game $\mathbb{G}$ is equivalent to the single-item single-bidder Myerson auction with the allocation rule $x(b) = P_{succ}(\pi_{opt(b)})$ and the payment rule $y(b) = E(\pi_{opt(b)}) \cdot c$, where b is the bidding value, $\pi_{opt(R)}$ denotes the optimal gacha pulling policy for a buyer with valuation R at initial state S_1, which is formulated as*

$$opt(R) = \begin{cases} a_0 = 0, & R \le \frac{E(\pi_{a_1})\cdot c}{P_{succ}(\pi_{a_1})}, \\ a_i, & \frac{(E(\pi_{a_i}) - E(\pi_{a_{i-1}}))\cdot c}{P_{succ}(\pi_{a_i}) - P_{succ}(\pi_{a_{i-1}})} < R \le \frac{(E(\pi_{a_{i+1}}) - E(\pi_{a_i}))\cdot c}{P_{succ}(\pi_{a_{i+1}}) - P_{succ}(\pi_{a_i})}, \\ a_L = \infty, & R > \frac{(E(\pi_\infty) - E(\pi_{a_{L-1}}))\cdot c}{P_{succ}(\pi_\infty) - P_{succ}(\pi_{a_{L-1}})}, \end{cases}$$

$P_{succ}(\pi_k) = 1 - \prod_{j=1}^{k}(1 - p_j)$ denotes the probability of winning the gacha game with policy π_k, and $E(\pi_k)$ denotes the expected number of gacha pulls with policy π_k, which is formulated in Lemma 3.3.

Proof. According to the definition of the whale property subgame in Definition 4.6, the optimal policy for a buyer with his personal valuation $R \in \mathbb{R}^+$ at the initial state S_1 could be $\pi_{a_i}, i = 0, 1, \cdots, L$. Define $\text{opt}(R) \in \{0, 1, 2, \cdots\}$ is a function of the buyer's valuation R, and $\pi_{\text{opt}(R)}$ denotes the optimal gacha pulling policy for a buyer with valuation R at the initial state S_1. According to Lemma B.1, the function $\text{opt}(R)$ is monotone. Therefore, if the buyer's optimal policy is π_{a_i}, his valuation R should satisfy the following conditions:

$$\begin{cases} V_{\pi_{a_i}}(S_1, R) \le V_{\pi_{a_{i+1}}}(S_1, R), & i = 0, \\ V_{\pi_{a_{i-1}}}(S_1, R) < V_{\pi_{a_i}}(S_1, R) \le V_{\pi_{a_{i+1}}}(S_1, R), & i = 1, 2, \cdots, L-1, \\ V_{\pi_{a_i}}(S_1, R) > V_{\pi_{a_{i-1}}}(S_1, R), & i = L. \end{cases} \tag{10}$$

For convenience, we denote the probability of winning the gacha game with policy π_k as $P_{\text{succ}}(\pi_k)$, which is calculated as follows:

$$P_{\text{succ}}(\pi_k) = 1 - \prod_{j=1}^{k}(1 - p_j).$$

Then combining Lemma 3.1 and Lemma 3.3, the value of MDP for the buyer with valuation R and policy π_k at initial state S_1 can be simplified as

$$\begin{aligned} V_{\pi_k}(S_1, R) &= \left(1 - \prod_{j=1}^{k}(1 - p_j)\right) R - \left(p_1 + \sum_{m=2}^{k} m p_m \prod_{i=1}^{m-1}(1 - p_i) + k \prod_{i=1}^{k}(1 - p_i)\right) c \\ &= P_{\text{succ}}(\pi_k) R - E(\pi_k) \cdot c, \end{aligned}$$

where $E(\pi_k)$ denotes the expected number of gacha pulls with policy π_k. Thus, the value of MDP for the buyer with valuation R and policy π_{a_i} at initial state S_1 is

$$V_{\pi_{a_i}}(S_1, R) = P_{\text{succ}}(\pi_{a_i})R - E(\pi_{a_i}) \cdot c$$

Then by solving the inequality (10), the optimal policy $\pi_{\text{opt}(R)}$ for the buyer with valuation R at initial state S_1 is

$$\text{opt}(R) = \begin{cases} a_0 = 0, & R \le \frac{E(\pi_{a_1}) \cdot c}{P_{\text{succ}}(\pi_{a_1})}, \\ a_i, & \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{\text{succ}}(\pi_{a_i}) - P_{\text{succ}}(\pi_{a_{i-1}})} < R \le \frac{\left(E(\pi_{a_{i+1}}) - E(\pi_{a_i})\right) \cdot c}{P_{\text{succ}}(\pi_{a_{i+1}}) - P_{\text{succ}}(\pi_{a_i})}, \\ a_L = \infty, & R > \frac{\left(E(\pi_\infty) - E(\pi_{a_{L-1}})\right) \cdot c}{P_{\text{succ}}(\pi_\infty) - P_{\text{succ}}(\pi_{a_{L-1}})} \end{cases} \tag{11}$$

Then the value of the buyer with valuation R and his optimal policy $\pi_{\text{opt}(R)}$ at initial state S_1 is

$$V_{\pi_{\text{opt}(R)}}(S_1, R) = P_{\text{succ}}(\pi_{\text{opt}(R)}) \cdot R - E(\pi_{\text{opt}(R)}) \cdot c \tag{12}$$

We now design a single-item single-bidder Myerson auction and then show its equivalence to the gacha game. We first design the allocation rule with bidding value b as follows:

$$x(b) = \begin{cases} P_{\text{succ}}(\pi_{a_0}) = 0, & b \le \frac{E(\pi_{a_1}) \cdot c}{P_{\text{succ}}(\pi_{a_1})}, \\ P_{\text{succ}}(\pi_{a_i}) \in (0,1), & \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{\text{succ}}(\pi_{a_i}) - P_{\text{succ}}(\pi_{a_{i-1}})} < b \le \frac{\left(E(\pi_{a_{i+1}}) - E(\pi_{a_i})\right) \cdot c}{P_{\text{succ}}(\pi_{a_{i+1}}) - P_{\text{succ}}(\pi_{a_i})}, \\ P_{\text{succ}}(\pi_{a_L}) = 1, & b > \frac{\left(E(\pi_\infty) - E(\pi_{a_{L-1}})\right) \cdot c}{P_{\text{succ}}(\pi_\infty) - P_{\text{succ}}(\pi_{a_{L-1}})} \end{cases} \tag{13}$$

Since $a_{i-1} < a_i, i = 1, 2, \cdots, L$ and $P_{\text{succ}}(\pi_k)$ is monotonically increasing on k, we know that the allocation rule $x(b)$ is also monotone. According to Myerson's Lemma, we know that the allocation rule $x(b)$ is also implementable, and there exists a unique payment rule $y(b)$ such that the sealed-bid auction mechanism (x, y) is dominant-strategy incentive-compatible (DSIC). The payment rule $y(b)$ is calculated as follows:

$$\begin{aligned} y(b) &= \int_0^b z \frac{d}{dz} x(z) dz \\ &= b \cdot x(b) - \int_0^b x(z) dz \\ &= \begin{cases} 0, & b \le \frac{E(\pi_{a_1}) \cdot c}{P_{\text{succ}}(\pi_{a_1})}, \\ \sum_{j=1}^{i} \left(P_{\text{succ}}(\pi_{a_i}) \cdot \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{\text{succ}}(\pi_{a_i}) - P_{\text{succ}}(\pi_{a_{i-1}})} \right), & \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{\text{succ}}(\pi_{a_i}) - P_{\text{succ}}(\pi_{a_{i-1}})} < b \le \frac{\left(E(\pi_{a_{i+1}}) - E(\pi_{a_i})\right) \cdot c}{P_{\text{succ}}(\pi_{a_{i+1}}) - P_{\text{succ}}(\pi_{a_i})}, \\ \sum_{j=1}^{L} \left(P_{\text{succ}}(\pi_{a_i}) \cdot \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{\text{succ}}(\pi_{a_i}) - P_{\text{succ}}(\pi_{a_{i-1}})} \right), & b > \frac{\left(E(\pi_\infty) - E(\pi_{a_{L-1}})\right) \cdot c}{P_{\text{succ}}(\pi_\infty) - P_{\text{succ}}(\pi_{a_{L-1}})} \end{cases} \\ &= \begin{cases} E(\pi_{a_0}) \cdot c = 0, & b \le \frac{E(\pi_{a_1}) \cdot c}{P_{\text{succ}}(\pi_{a_1})}, \\ E(\pi_{a_i}) \cdot c, & \frac{\left(E(\pi_{a_i}) - E(\pi_{a_{i-1}})\right) \cdot c}{P_{\text{succ}}(\pi_{a_i}) - P_{\text{succ}}(\pi_{a_{i-1}})} < b \le \frac{\left(E(\pi_{a_{i+1}}) - E(\pi_{a_i})\right) \cdot c}{P_{\text{succ}}(\pi_{a_{i+1}}) - P_{\text{succ}}(\pi_{a_i})}, \\ E(\pi_{a_L}) \cdot c, & b > \frac{\left(E(\pi_\infty) - E(\pi_{a_{L-1}})\right) \cdot c}{P_{\text{succ}}(\pi_\infty) - P_{\text{succ}}(\pi_{a_{L-1}})} \end{cases} \end{aligned} \tag{14}$$

With the payment rule $y(b)$, the mechanism is DSIC and thus truthful. Therefore, the bidding value b is exactly the buyer's personal valuation R. Comparing equation (11) and (13), we can find that $x(b) = P_{\text{succ}}(\pi_{\text{opt}(b)})$. Comparing equation (11) and (14), we can find that $y(b) = E(\pi_{\text{opt}(b)}) \cdot c$. Following the quasilinear utility model of the bidder in Myerson auction, the bidder's utility is

$$\begin{aligned} u(R) &= R \cdot x(R) - y(R) \\ &= P_{\text{succ}}(\pi_{\text{opt}(R)}) \cdot R - E(\pi_{\text{opt}(R)}) \cdot c \end{aligned}$$

which is exactly the MDP value of the buyer with valuation R and his optimal policy $\pi_{\mathrm{opt}(R)}$ at initial state S_1 in equation (12). Therefore, the gacha game is equivalent to the Myerson auction with the stochastic allocation rule $x(b)$ and the payment rule $y(b)$. The proof is thus completed. □

Theorem 4.10 *The optimal gacha game that can achieve the maximum seller's revenue should have whale property and satisfy the following condition:*

$$E(\pi_\infty) \cdot c = r^* = \arg\max_r r \cdot (1 - F(r)),$$

where c is the cost of each gacha pull, $E(\pi_\infty)$ is the expected number of gacha pulls to win the gacha game with policy π_∞, r^* is the optimal reserved price in the single-item single-bidder Myerson auction.

Proof. According to Theorem 4.8 and Theorem 4.9, we know that only the whale property gacha game can achieve the maximum seller's revenue. To prove the theorem above, we only need to prove that in the gacha game with the whale property, for a buyer with valuation R, where R follows the distribution F, when $R \geq cE(\pi_\infty)$, we have $V_{\pi_\infty}(S_1) \geq 0$. This implies that when the valuation is greater than or equal to $cE(\pi_\infty)$, the optimal policy for the buyer is π_∞, that is, the buyer will continue pulling the gacha until he wins the gacha game.

According to Lemma 3.1, the value at the initial state S_1 in the MDP is

$$V_{\pi_\infty}(S_1) = R - \left(p_1 + \sum_{m=2}^{\infty} m p_m \prod_{i=1}^{m-1} (1 - p_i)\right) c.$$

Therefore, the optimal policy for the buyer is π_∞ when

$$V_{\pi_\infty}(S_1) \geq 0 \Leftrightarrow R \geq \left(p_1 + \sum_{m=2}^{\infty} m p_m \prod_{i=1}^{m-1} (1 - p_i)\right) c = E(\pi_\infty).$$

According to the analysis above, we know that when $R \geq cE(\pi_\infty)$, we have $V_{\pi_\infty} \geq 0$. Therefore, to maximum the revenue, the seller needs to carefully design the game configuration such that

$$\arg\max_{c,\pi} cE(\pi_\infty) \cdot (1 - F(cE(\pi_\infty)))\,.$$

Therefore, when

$$cE(\pi_\infty) = r^* = \arg\max_r r \cdot (1 - F(r)),$$

the gacha game with the whale property can achieve the maximum revenue and r^* is exactly the optimal mechanism design for the single-item single-bidder Myerson auction. The proof is thus completed. □

Proposition 1 *With budget constraints, the whale property gacha game can achieve a higher seller's revenue than the "take-it-or-leave-it" strategy.*

Proof. We first show that the maximum seller's revenue in the whale property gacha game is at least greater than or equal to that in the "take-it-or-leave-it" strategy. Assume that the optimal price that can achieve the maximum seller's revenue in the "take-it-or-leave-it" strategy is r^*. Then construct the gacha game $\mathbb{G}$ with the price of the gacha pull c, where r^*/c is an integer, and the probability that $p_i = 0, \forall i < r^*/c$ and $p_{r^*/c} = 1$, the gacha game $\mathbb{G}$ can achieve the same seller's revenue in the "take-it-or-leave-it" strategy is r^*. Combing Example 1, we can find that with budget constraints, the whale property gacha game can achieve a higher seller's revenue than the "take-it-or-leave-it" strategy.

Suppose that each buyer's budget is a realization of the random variable B with distribution F_B. Let the joint probability density function of the budget B and the valuation R be $f(R, B)$. The revenue of the "take-it-or-leave-it" selling strategy with price r is

$$U_{\text{TILI}}(r) = r \cdot \int_r^{\infty} \int_r^{\infty} f(R, B) dR dB.$$

Here we consider a simple fixed-probability gacha game with winning probability p and the price of gacha pull c. In the gacha game, the buyers will purchase the gacha pull until they win the gacha game or until they run out of their budget. Suppose that the equivalent price for winning the gacha game is r, then the seller's revenue is

$$\begin{aligned} U_{\text{Gacha}}(p, c) &= \int_0^{\infty} E(\pi_{\lfloor \frac{B}{c} \rfloor}) \cdot c \int_{\frac{c}{p}}^{\infty} f(R, B) dR dB \\ &= \int_0^{\infty} \frac{1 - (1-p)^{\lfloor \frac{B}{c} \rfloor}}{p} \cdot c \int_{\frac{c}{p}}^{\infty} f(R, B) dR dB \end{aligned}$$

Due to robustness to the random fluctuations in buyer's budge, the gacha game with the whale property is possible to achieve a higher seller's revenue when the buyers are budget-constrained. □

C PROOFS IN SECTION 5

Proposition 2 *For the whale property sequential multi-item gacha game with the reset-after-winning mechanism, the buyer will continue pulling the gacha game until he has won k^* times, where*

$$k^* = \arg \max_{k=0,1,2,\cdots,K} \left\{ \left(\sum_{j=1}^{k} R_j \right) - kE(\pi_\infty) \cdot c \right\}. \tag{15}$$

Specially, $k^* = 0$ implies that the buyer will never pull the gacha game. Here $E(\pi_\infty)$ denotes the expected number of gacha pulls needed to win the gacha game once, which is formulated in Lemma 3.3, and c denotes the price of each gacha pull.

Proof. The whale property guarantees that the buyer would not pull the gacha and stop midway without winning the gacha game. Therefore, the buyer will only stop when he has won the gacha game with $0, 1, 2, \cdots, K$ times. To figure out the optimal policy that maximizes his utility, the buyer needs to decide when to stop, which can be considered as an optimal stopping problem. Besides, the reset-after-winning mechanism guarantees that the expected cost for each win of the gacha game remains the same, i.e., $E(\pi_\infty) \cdot c$. If the buyer wants to stop when he has won the gacha game with k times, his expected profit will be $\left\{ \left(\sum_{j=1}^{k} R_j \right) - kE(\pi_\infty) \cdot c \right\}$. The buyer will figure out the optimal k^* to maximize his profit, which is listed in (15).

Besides, we will show that the optimal k^* always holds during the buyer's gacha pulling process. Initially, the buyer never wins the gacha game before, then he will figure out the optimal k^* to maximize his profit, which is listed in (15). During the buyer's gacha pulling process, suppose that the buyer has won the gacha game t times ($t < k^*$). Then the buyer will continue pulling the gacha game until he has won more $m^*(t)$ times, where

$$m^*(t) = \arg \max_{m=0,1,2,\cdots,K-t} \left\{ \left(\sum_{j=t+1}^{t+m} R_j \right) - mE(\pi_\infty) \cdot c \right\}.$$

We will show that $k^* \equiv t + m^*(t)$. We will prove it by contradiction. Suppose that $t + m^*(t) \neq k^*$. Then we have

$$\begin{aligned} k' &= \left(\sum_{j=1}^{t} R_j\right) - tE(\pi_\infty)\cdot c + \arg\max_{k=t+1,t+2,\cdots,K} \left\{\left(\sum_{j=t+1}^{k} R_j\right) - kE(\pi_\infty)\cdot c\right\} \\ &= \left(\sum_{j=1}^{t} R_j\right) - tE(\pi_\infty)\cdot c + \arg\max_{m=0,1,2,\cdots,K-t} \left\{\left(\sum_{j=t+1}^{t+m} R_j\right) - mE(\pi_\infty)\cdot c\right\} \\ &= t + m^*(t) \end{aligned}$$

Since the buyer has won the gacha game t times ($t < k^*$), we know $k^* = k'$, which leads to contradiction. Therefore, during the mining process, the buyer will continue pulling the gacha game until he has won k^* times. The proof is thus completed. □

Proposition 3 *For the whale property sequential multi-item gacha game with the succeed-after-winning mechanism, the buyer that has won the gacha game k times ($k = 0, 1, \cdots, K-1$), will pull the gacha at state S_i if and only if*

$$\max_{t=1,2,\cdots,K-k} \left\{\left(\sum_{j=k+1}^{k+t} R_j\right) - H(t,i)\cdot c\right\} \geq 0,$$

where $H(t,i)$ denotes the expected number of gacha pulls needed to win the gacha game t more times when the buyer is at state S_i, which can be recursively calculated as follows:

$$H(t,i) = \begin{cases} \sum_{j=i}^{\infty} p_j \prod_{t=i}^{j-1}(1-p_t)\cdot(j-i+1), & t = 1, \\ \sum_{j=i}^{\infty} p_j \prod_{t=i}^{j-1}(1-p_t)\cdot(j-i+1+H(t-1,j+1)), & t > 1. \end{cases}$$

where p_i is the probability to win the gacha game at state S_i.

Proof. The whale property guarantees that the buyer would not pull the gacha and stop midway without winning the gacha game. Therefore, the buyer will only stop when he has won the gacha game with $0, 1, 2, \cdots, K$ times. To figure out the optimal policy to maximize his utility, the buyer needs to decide when to stop, which is an optimal stopping problem. If the buyer stops pulling the gacha anymore, he will get 0 profit at state S_i. On the other hand, if the buyer wants $t, 1 \leq t \leq K-k$ more wins, his profit will be $\left(\sum_{j=k+1}^{k+t} R_j\right) - H(t,i)\cdot c$, where $H(t,i)$ denotes the expected number of gacha pulls needed to win the gacha game t times when the buyer is at state S_i. Since the buyer is currently at state S_i, the probability that the buyer wins the gacha game at state $S_j, j = i, i+1, \cdots, N$ is $(p_j \prod_{t=i}^{j-1}(1-p_t))$. Once the buyer wins the gacha at state S_j, the buyer will enter the next state S_{j+1}. Besides, the buyer only needs $t-1$ more wins if he has won at state S_j. Therefore, we have (2). The proof is thus completed. □

Theorem 5.1 *For the sequential multi-item gacha game with infinite items, and the buyer's valuation of each item follows the identical and independent distribution with mean μ and variance σ^2, the whale property sequential multi-item gacha game with the reset-after-winning mechanism can achieve the asymptotic optimality on seller's revenue, i.e.,*

$$\lim_{K\to\infty} \frac{c\cdot\mathbb{E}(\text{\# of gacha pulls purchases})}{K} = \mu,$$

where $\frac{c\cdot\mathbb{E}(\text{\# of gacha pulls purchases})}{K}$ denotes the normalized seller's revenue, c is the price of each gacha pull and K is the number of items in the gacha game.

Proof. **sketch**: Assume that the buyer has won the gacha game k times. When K is large, while k is relatively small, then the low of large numbers implies that $\frac{1}{K-k}\left(\sum_{j=k+1}^{K} R_j\right) \approx \mu$. Therefore, if the price of this gacha game is $E \cdot c = \mu - \epsilon$, where ϵ can be arbitrarily small, the buyer will pull the gacha game until he wins again according to Proposition 2. Recursively, the buyer will continue pulling the gacha until he has won $K - T$ times and the expected valuation for the remaining gacha game $\frac{1}{K-T}\left(\sum_{j=T}^{K} R_j\right)$ is smaller than $\mu - \epsilon$. Since K is large, while T is relatively small, $T/K \longrightarrow 0$. Therefore, the buyer will pull the gacha almost all times with price for each time being $\mu - \epsilon$, which implies the asymptotic optimality on seller's revenue.

detail: Let v_j denote the user's valuation for winning the gacha game at j-th time, $x_k = \frac{1}{k}\sum_{j=K-k+1}^{K} v_j$ denote the normalized expected valuation for bundling the reward of the gacha game winning at $j = K-k, K-k+1, \cdots, K$ time. Let $\mu_k = \mathbb{E}(x_k)$ and $\sigma_k = \mathbb{E}\left(|x_k - \mu_k|^2\right)$. Let $\lim_{k\to\infty} \mu_k = \mu$ and $\lim_{k\to\infty} \sigma_k = \sigma$. Denote by p_k^*, q_k^* the optimal price $(E \cdot c)$ for the gacha game and the corresponding quantity $(0 \le q_l^* \le 1)$, and let π_k^* be the resulting profits $\pi_k^* = p_k^* q_k^*$. Let $\lim_{k\to\infty} p_k^* = P$ and $\lim_{k\to\infty} q_k^* = Q$. We will show that $P = \mu$ and $Q = 1$.

If $P > \mu$, there exists some $\epsilon > 0$ such that for all large enough k, $p_k^* > \mu + \epsilon$. By the weak law of large numbers, we know that $\Pr\left(|x_k - \mu| < \epsilon\right) \ge 1 - \delta$, where $n \ge \frac{\delta^2}{\epsilon^2\delta}$ or $\delta \le \frac{\sigma^2}{\epsilon^2 n}$. Thus, if $P > \mu$, $\{q_k^*\} \longrightarrow 0$, and since $\{p_k^*\}$ is bounded, we have $\{\pi_k^*\} \longrightarrow 0$, which contradicts the optimality of p_k^* and q_k^*.

If $P < \mu$, there exists some $\epsilon > 0$ such that for all large enough k, $p_k^* < \mu - \epsilon$. Let $\hat{p_k} = P + \frac{\epsilon}{2}$, and $\hat{q}$ the corresponding quantity. The weak law of large numbers implies that $\lim_{k\to\infty} q_k^* = \lim_{k\to\infty} \hat{q_k} = 1$ and $\lim_{k\to\infty}(q_k^* - \hat{q_k}) = 0$. Since for large enough k, $\hat{p_k} - p_k^* \ge \frac{\epsilon}{2}$, it follows that $\hat{p_k}\hat{q_k} > p_k^* q_k^*$, which again contradicts the optimality of p_k^* and q_k^*. Thus, $\lim_{k\to\infty} p_k^* = \mu$. Then we can show $P = \mu$ and $Q = 1$, which implies the asymptotic optimality on seller's revenue. The proof is thus completed. □

Theorem 5.2 *The banner-based gacha game with the reset-after-opt-out mechanism, the banner-based gacha game with the succeed-after-opt-out mechanism, and the separate selling with several independent single-item gacha games are equivalent, i.e., the behaviors of the rational buyers and the seller's revenues in these gacha game are the same.*

Proof. We first show that the multi-item gacha game with two banners with values of R_1 and R_2 is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$. And the theorem above can be obtained by induction.

It is easy to prove that when $R_1 \ge R_2$, it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$. The reason goes as follows. On the one hand, if the player will pull in both $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$, he will also pull the gacha in both banners. On the other hand, if the player will only pull in $\mathbb{G}(R_1)$ and will not pull in $\mathbb{G}(R_2)$, the player will pull the first banner and will not stop pulling the gacha in the first banner and wait for the second banner since $R1 \ge R_2$. Therefore, when $R_1 \ge R_2$ (Scenario 1), it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$.

We next consider the scenario where $R_1 < R_2$, and show that it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$. We consider the three cases:

- **Case 1**: The player will not pull either $\mathbb{G}(R_1)$ or $\mathbb{G}(R_2)$.
- **Case 2**: The player will pull both $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$.
- **Case 3**: The player will only pull $\mathbb{G}(R_2)$ but will not pull in $\mathbb{G}(R_1)$.

It is obvious that in **Case 1** the player will not pull in the multi-items gacha game in any banners, because pulling gacha in any banner will leave him with negative utility. Thus, in this case, it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$.

We next investigate the **Case 2**, and further show that it is not profitable for the player to stop pulling the gacha in the first banner and wait for the second banner. Thus, we can prove that in this case, it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$. The proof goes as follows.

We are going to prove by contradiction. Suppose that the player that has pulled $n-1$ gacha pulls will stop pulling the gacha in the first banner and wait for the second banner. The player's utility that he will pull in the n-th round, but will end in the $(n+1)$-th round in $\mathbb{G}(R_1)$ is analyzed as follows. The probability that the player wins the gacha game $\mathbb{G}(R_1)$ in the (n)-round is p_n. Once he wins $\mathbb{G}(R_1)$, he will wait and enter the game $\mathbb{G}(R_2)$ with his state being S_1. On the other hand, if he does not win the game $\mathbb{G}(R_1)$ in the n-th round, he will wait and enter the game $\mathbb{G}(R_2)$ with his state being S_n. Then we have

$$\begin{aligned} &V_{\pi_\infty}(S_n, R_2) > V_{\pi_\infty}(S_n, R_1) + V_{\pi_\infty}(S_1, R_2) \\ \Rightarrow &R_2 - E_n c > R_1 - E_n c + R_2 - Ec \\ \Rightarrow &R_1 - Ec < 0 \end{aligned} \tag{16}$$

where E denotes the expected number of gacha pulls needed to win the gacha game at state S_1 and $E_n = 1 + \sum_{m=n+1}^{\infty} \prod_{j=n}^{m-1}(1-p_j)$ denotes the expected number of gacha pulls needed to win the gacha game at state S_n. $R_1 - Ec = v_{a_1}(R, 1) < 0$ implies that the player will not pull the gacha game $\mathbb{G}(R_1)$, which leads to contradiction. Therefore, the player will not stop pulling the gacha in the first banner. Obviously, after winning the gacha game in the first banner, the gacha game in the second banner is equivalent to the independent single-item gacha game $\mathbb{G}(R_2)$. Therefore, in this case, it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$.

We next investigate the **Case 3**, and further show that it is not profitable for the player to pull a small number of gacha pulls in the first banner and wait for the second banner. Similarly, we will show that if the player stops pulling the gacha game in the first banner in any round, he would never pull the gacha in the first banner. Suppose that the player that has pulled $n-1$ gacha pulls will stop pulling the gacha in the first banner and wait for the second banner. Then we have

$$\begin{aligned} &p_n(R_1 + V_{\pi_\infty}(S_1, R_2)) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_2) > V_{\pi_\infty}(S_n, R_2) \\ \Rightarrow &p_n(R_1 + R_2 - Ec) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_2) > p_n R_2 - c + (1-p_n)V(S_{n+1}, R_2) \\ \Rightarrow &R_1 - Ec > 0 \end{aligned}$$

where E denotes the expected number of gacha pulls needed to win the gacha game. $R_1 - Ec = V_{\pi_\infty}(S_1, R_1) > 0$ implies that the player will pull the gacha game $\mathbb{G}(R_1)$, which leads to contradiction. Therefore, the player will never pull the gacha in the first banner, and then the gacha game in the second banner is equivalent to the independent single-item gacha game $\mathbb{G}(R_2)$. Therefore, in this case, it is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$.

To sum up, we can claim that the multi-item gacha game with two banners with values of R_1 and R_2 is equivalent to the single-item gacha games as $\mathbb{G}(R_1)$ and $\mathbb{G}(R_2)$. And the theorem above can be obtained by induction. The proof is thus completed. □

D PROOFS IN SECTION 6

Theorem 6.1 *The subsidies in fixed-probability gacha game always degrade the seller's revenue compared to the gacha game without any subsidies.*

Proof. In the fixed-probability gacha game with m free gacha pulls, the buyers always pull the subsidized m free gacha pulls first. If the buyers do not win the gacha game within m gacha pulls, the buyers with high valuation, i.e., $R \geq c/p$ will buy the gacha pull from the seller to continue pulling the gacha game, and those with low valuation, i.e., $R < c/p$ will stop and leave the gacha game. Therefore, the optimal policy for the buyer is either π_m or π_∞. For the buyer with high valuation $R \geq c/p$, the probability that the buyer wins the gacha game within m gacha pulls is $(1-(1-p)^m)$, and the expected number of free gacha pulls spent in the gacha game is

$$E_s(\pi_\infty, m) = \sum_{n=1}^{m} np(1-p)^{n-1} = \frac{1-(1-p)^m}{p} - m(1-p)^m.$$

The probability that the buyer spends all the free gacha pulls and needs to buy the gacha pull is $(1-p)^m$, and the expected number of the bought gacha pulls in the gacha game is

$$\begin{aligned} E_b(\pi_\infty, m) &= \sum_{n=m+1}^{\infty} (n-m)p(1-p)^{n-1} \\ &= \sum_{n=m+1}^{\infty} np(1-p)^{n-1} - m \sum_{n=m+1}^{\infty} p(1-p)^{n-1} \\ &= \frac{(1-p)^m}{p} + m(1-p)^m - m(1-p)^m = \frac{(1-p)^m}{p}. \end{aligned}$$

The seller can only obtain revenue from the buyers' bought gacha pulls. Therefore, the seller's revenue with subsidies in fixed-probability gacha game is

$$\begin{aligned} U_s(c,p,m) &= E_b(\pi_\infty, m)c \cdot \left(1 - F(\frac{c}{p})\right) \\ &= \frac{(1-p)^m}{p} \cdot c \cdot \left(1 - F(\frac{c}{p})\right) < \frac{c}{p}\left(1 - F(\frac{c}{p})\right), \end{aligned}$$

where the $\frac{c}{p}\left(1 - F(\frac{c}{p})\right)$. The proof is thus completed. □

Theorem 6.2 *For a whale property gacha game, where the winning probability at state S_i is p_i and the cost for each gacha pull is c, if m free gacha pulls are subsidized in this gacha game, only the buyers with valuations greater than $\varphi_s(m) = \left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=m}^{i-1}(1-p_j)\right) \cdot c$ will buy the gacha pull when they run out all the free gacha pulls, and the seller's revenue with m free gacha pulls is*

$$U_s(m) = \left(\sum_{i=m+1}^{\infty} (i-m)p_i \prod_{j=1}^{i-1}(1-p_j)\right) \cdot c \cdot \left(1 - F\left(\left(\sum_{i=m+1}^{\infty} (i-m)p_i \prod_{j=m+1}^{i-1} (1-p_j)\right) \cdot c\right)\right).$$

When $\arg\max U_s(m) > 0$, *subsidies can improve the seller's revenue.*

Proof. The expected number of free gacha pulls spent in the gacha game for buyers is

$$E_s(\pi_m, m) = \sum_{i=1}^{m} ip_i \prod_{j=1}^{i-1}(1-p_j).$$

If the buyers do not win the gacha game within m free gacha pulls, the buyer will be at state S_{m+1}. The whale property guarantees that the optimal policy for the buyer is either π_∞ and π_m. The value for the buyer with valuation R at state S_{m+1} is

$$\begin{aligned} V_{\pi_\infty}(S_{m+1}) &= \left(p_{m+1} + \sum_{j=m+2}^{\infty} p_j \prod_{t=m+1}^{j-1} (1-p_t)\right) R - \left(1 + \sum_{j=m+2}^{\infty} \prod_{t=m+1}^{j-1} (1-p_t)\right) c \\ &= R - \left(\sum_{i=m+1}^{\infty} (i-m)p_i \prod_{j=m+1}^{i-1} (1-p_j)\right) \cdot c. \end{aligned}$$

Therefore, only the buyers with high valuation, i.e., $R \geq \varphi_s(m) = \left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=m}^{i-1}(1-p_j)\right) \cdot c$ will buy and continue pulling the gacha game. Besides, the expected number of bought gacha pulls in the gacha game is

$$E_b(\pi_\infty, m) = \left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=1}^{i-1}(1-p_j)\right).$$

Therefore, the seller's revenue with subsidies in the whale property gacha game is

$$\begin{aligned} U_s(m) &= E_b(\pi_\infty, m)c \cdot \left(1 - F\left(\left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=m+1}^{i-1}(1-p_j)\right) \cdot c\right)\right) \\ &= \left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=1}^{i-1}(1-p_j)\right) \cdot c \cdot \left(1 - F\left(\left(\sum_{i=m+1}^{\infty}(i-m)p_i \prod_{j=m+1}^{i-1}(1-p_j)\right) \cdot c\right)\right). \end{aligned}$$

The proof is thus completed. □

Theorem 6.3 *Consider a banner-based multi-item gacha game where there are K banners, and the buyer's valuations on the K items in these banners are $R_1, R_2, \ldots, R_K$. Suppose the buyer is currently at the i-th banner with m accumulated free gacha pulls subsidized by the seller. There are three possible scenarios:*

- *If $R_i \geq E(\pi_\infty) \cdot c$, the buyer will use his free gacha pulls to pull the gacha game in this banner, and if he uses out all the free gacha pulls, he will buy the gacha pull and pull it until he wins in this banner.*
- *If $R_i < E(\pi_\infty) \cdot c$, and there exists a banner $j (i < j \leq K)$, such that $R_j \geq E(\pi_\infty) \cdot c$, then the buyer would not buy any gacha pull and pull it before the j-th banner. The buyer will use the free gacha pulls to pull in the j-th banner until he wins. After exhausting all the free gacha pulls, he will buy the gacha pulls and use them to pull in the gacha game until he wins.*
- *If $R_j < E(\pi_\infty) \cdot c, \forall j \in [i, K]$, let $k^* = \arg\max_{k \in [i,K]} R_k$, then the buyer would not buy any gacha pull and pull it before the k^*-th banner, and uses the free gacha pulls to pull in the k^*-th banner until he wins. If $R_{k^*} < E(\pi_\infty, S_{m+1}) \cdot c$, when the buyer uses out all the free gacha pulls in the k^*-th banner, he will stop pulling the gacha. Otherwise, $R_{k^*} \geq E(\pi_\infty, S_{m+1}) \cdot c$, when the buyer uses out all the free gacha pulls in the k^*-th banner, he will buy the gacha pull and pull it until he wins in this banner.*

Here $E(\pi_\infty)$ denotes the expected number of gacha pulls needed to win the gacha game, and $E(\pi_\infty, S_{m+1})$ denotes the expected number of gacha pulls needed to win the gacha game when the buyer is at state S_{m+1}, which is formulated in Lemma 3.3.

Proof. For the first case where $R_i \geq E(\pi_\infty) \cdot c$, the buyer will pull the gacha even without the subsidies. So the buyer will use his free gacha pulls to pull the gacha game in this banner, and if he uses out all the free gacha pulls, he will buy the gacha pull and pull it until he gets the desired item in this banner.

For the second case where $R_i < E(\pi_\infty) \cdot c$, and there exists a banner $j (i < j \leq K)$, such that $R_j \geq E(\pi_\infty) \cdot c$, we will first prove that the buyer will not pull the gacha in the n-th round in the $j-1$-th banner when $n > m$, which implies that the buyer will not buy the gacha pull in the $j-1$-th banner. Then we will prove that by mathematical introduction, the buyer will not buy any gacha pull before the j-th banner.

We have that $R_{j-1} < E(\pi_\infty) \cdot c$, therefore we have

$$R_{j-1} - c + R_j - E(\pi_\infty) \cdot c < R_j - c,$$

which implies that in the $j-1$-th banner, the buyer will not buy the gacha in N-th round. Then we are going to show that if the buyer will not buy the gacha in the $(n+1)$-th round, he will also not buy the gacha in n-th round ($n > m$). Equivalently, we need to prove the following inequality:

$$\begin{aligned}
&p_n(R_{j-1} + V_{\pi_\infty}(S_1, R_j)) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_j) < V_{\pi_\infty}(S_n, R_j) \\
\Rightarrow &p_n(R_{j-1} + V_{\pi_\infty}(S_1, R_j)) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_j) < p_n R_j - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_j) \\
\Rightarrow &p_n(R_{j-1} + R_j - E(\pi_\infty)c) < p_n R_j \\
\Rightarrow &R_{j-1} - E(\pi_\infty)c < 0
\end{aligned}$$

By mathematical induction, we can prove that the buyer will not buy the gacha in the (n)-th round $\forall n > m$. Then in the $j-2$-th banner, similarly we have

$$\begin{aligned}
&p_n(R_{j-2} + V_{\pi_\infty}(S_1, R_j)) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_j) < V_{\pi_\infty}(S_n, R_j) \\
\Rightarrow &p_n(R_{j-2} + V_{\pi_\infty}(S_1, R_j)) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_j) < p_n R_j - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_j) \\
\Rightarrow &p_n(R_{j-2} + R_j - E(\pi_\infty)c) < p_n R_j \\
\Rightarrow &R_{j-2} - E(\pi_\infty)c < 0
\end{aligned}$$

Recursively, we can prove that the buyer would not buy any gacha pull and pull it before the j-th banner.

For the third case where $R_j < E(\pi_\infty) \cdot c, \forall j \in [i, K]$, if $R_{k^*} < E(\pi_\infty, S_{m+1}) \cdot c$, buy any gacha pull will leave the buyer's surplus being negative, thus the buyer will never buy any gacha pull. If $R_{k^*} > E(\pi_\infty, S_{m+1}) \cdot c$, we will first prove that the buyer will not pull the gacha in the n-th round in the $k^* - 1$-th banner when $n > m$, which implies that the buyer will not buy the gacha pull in the $k^* - 1$-th banner. Then we will prove that by mathematical introduction, the buyer will not buy any gacha pull before the k^*-th banner. We have that $R_{k^*-1} < R_{k^*}$, therefore we have

$$R_{k^*-1} - c < R_{k^*} - c,$$

which implies that in the $k^* - 1$-th banner, the buyer will not buy the gacha in N-th round. Then we are going to show that if the buyer will not buy the gacha in the $(n+1)$-th round, he will also not buy the gacha in n-th round ($n > m$). Equivalently, we need to prove the following inequality:

$$\begin{aligned}
&p_n(R_{k^*-1}) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_{k^*}) < V_{\pi_\infty}(S_n, R_{k^*}) \\
\Rightarrow &p_n(R_{k^*-1}) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_{k^*}) < p_n R_{k^*} - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_{k^*}) \\
\Rightarrow &R_{k^*-1} < R_{k^*}
\end{aligned}$$

By mathematical induction, we can prove that the buyer will not buy the gacha in the (n)-th round $\forall n > m$. Then in the $k^* - 2$-th banner, similarly we have

$$\begin{aligned}
&p_n(R_{k^*-2}) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_{k^*}) < V_{\pi_\infty}(S_n, R_{k^*}) \\
\Rightarrow &p_n(R_{k^*-2}) - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_{k^*}) < p_n R_{k^*} - c + (1-p_n)V_{\pi_\infty}(S_{n+1}, R_{k^*}) \\
\Rightarrow &R_{k^*-2} < R_{k^*}
\end{aligned}$$

Recursively, we can prove that the buyer would not buy any gacha pull and pull it before the j-th banner. The proof is thus completed. □

E DETAILED CALCULATIONS OF THE EXAMPLES

Example 1. There is a buyer who has the valuation of 100 and his budget B follows the distribution that $P(B = 50) = 0.5, P(B = 100) = 0.5$. The maximum seller's revenue achieved by the "take-it-or-leave-it" strategy is 50, whereas the fixed-probability gacha game with the probability being 0.01 and the price for each gacha pull being 1, can achieve the seller's revenue of 51.448.

PROOF. The optimal "take-it-or-leave-it" strategy is either to set the price to be 50, where the buyer will buy the item and the seller's revenue is 50, or to set the price to be 100, where only when the buyer has the budget of 100 would buy the item and the seller's revenue is also $100 * 0.5 = 50$.

For the fixed-probability gacha game with the probability being 0.01 and the price for each gacha pull being 1, the analysis goes as follows. Firstly, we will show that the buyer would pull the gacha game until he wins or his budget is exhausted. This is because pulling the gacha game provides a non-negative profit for the buyer, i.e. $pR - c \geq 0$. Then we know that when the buyer has the budget of 50, his optimal policy is π_{50}. According to Lemma 3.3, the expected number of gacha pulls the buyer buys is $E(\pi_{50}, S_1) = 39.499$. Similarly, when the buyer has the budget of 50, his optimal policy is π_{100}, and the expected number of gacha pulls the buyer buys is $E(\pi_{100}, S_1) = 63.397$. Therefore, the expected number of gacha pulls that the seller sells is $0.5 * E(\pi_{50}, S_1) + 0.5 * E(\pi_{100}, S_1) = 51.448$. With the price of each gacha pull being 1, the expected seller's revenue is 51.448, which is greater than that of the "take-it-or-leave-it" strategy. This demonstrates the efficiency of the gacha game when facing budget-constrained buyers. □

Example 2. Suppose that there are two items in the sequential gacha game, i.e., $K = 2$. The buyer's valuations for these two items are independently and identically distributed (i.i.d.), and follow the uniform distribution $[0, 1]$. By separately selling these two items at the same price, the maximum seller's revenue is 0.5. With the reset-after-winning mechanism, the maximum seller's revenue that the sequential gacha game can achieve is 0.516. While for the sequential gacha game with the succeed-after-winning mechanism and pity system where $N = 100, p_i = 0.172, \forall i < N$ and $P_N = 1$ and the price of the gacha pull $c = 0.01$, the seller's revenue is 0.5218.

PROOF. We first investigate the maximum seller's revenue achieved by separately selling these two items at the same price. Since the buyer's valuations on these two items are independently and identically distributed from the same distribution, we can turn to investigate the optimal pricing problem for one item. According to the single-item single-bidder Myerson auction, the seller's revenue with the reserved price r is

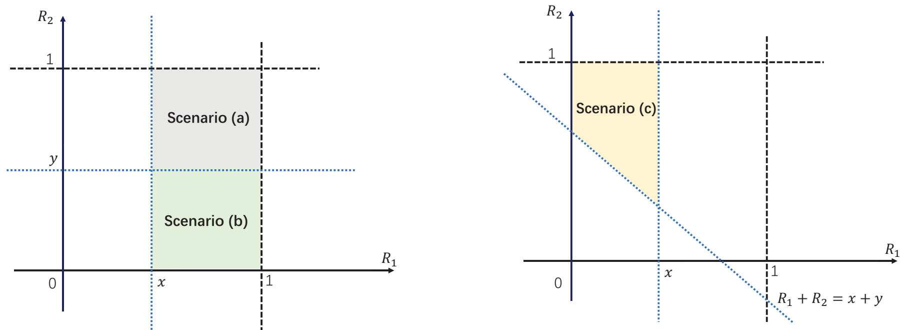
$$\text{Revenue}(r) = r \cdot (1 - r).$$

Thus, the optimal price is 0.5, and the maximum seller's revenue on one item is 0.25. Therefore, the maximum seller's revenue achieved by separately selling these two items is $(0.25 * 2 = 0.5)$, with the price of each item being 0.5.

We will next derive the optimal gacha game design with the reset-after-winning mechanism in the sequential gacha game. For convenience, we denote the buyer's valuation on the first item as R_1 and the buyer's valuation on the second item as R_2. Firstly, we consider the case where we sell these two items sequentially with the price of the first item being x and the price of the second item being y ($x \leq 1, y \leq 1$). Then there are the following possible scenarios:

(1) When $R_1 \geq x$ and $R_2 \geq y$, the buyer will buy the first item and the second item. In this case, the seller can obtain a revenue of $x + y$.
(2) When $R_1 \geq x$ but $R_2 < y$, the buyer will buy the first item and quit without buying the second item. In this case, the seller can obtain a revenue of x.
(3) When $R_1 < x$ but $R_1 + R_2 \geq x + y$, the buyer will buy the first item and the second item. In this case, the seller can obtain a revenue of $x + y$.
(4) Otherwise, the buyer will buy nothing, and the seller can obtain zero revenue.

Since R_1 and R_2 are independently and identically distributed from the uniform distribution $[0, 1]$, we can calculate the probability of each scenario. Figure 4 demonstrates the distribution area of each scenario. Because scenario (d) does not affect the seller's revenue, we only focus on the



(a) Distribution area of scenario (a) and (b)

(b) Distribution area of scenario (c)

Fig. 4. Distribution area of each scenario

probability of scenario (a), (b), (c), which is listed as follows:

$$\Pr((a)) = (1 - x) \cdot (1 - y),$$
$$\Pr((b)) = (1 - x) \cdot y,$$
$$\Pr((c)) = \begin{cases} (1 - x - y + 1 - y) * x/2, & x + y \leq 1, \\ (1 - y)^2/2, & \text{otherwise.} \end{cases}$$

Therefore, the seller's expected revenue with price x, y is

$$\begin{aligned} \text{Revenue}(x, y) &= \Pr((a)) \cdot (x + y) + \Pr((b)) \cdot x + \Pr((c)) \cdot (x + y) \\ &= (1 - x)(x + y(1 - y)) + \begin{cases} (2 - x - 2y)x/2 \cdot (x + y), & x + y \leq 1, \\ ((1 - y)^2)/2 \cdot (x + y), & \text{otherwise.} \end{cases} \end{aligned}$$

According to Theorem 4.10, we can consider the equivalent price of an item in the sequential gacha game with reset-after-winning mechanism as $E(\pi_\infty) \cdot c$, where $E(\pi_\infty)$ is the expected number of gacha pulls needed to win the game and c is the price of each gacha pull. In the sequential gacha game with the reset-after-winning mechanism, the equivalent prices of these two items are the same, i.e., $x \equiv y \equiv E(\pi_\infty) \cdot c$. Therefore, to figure out the optimal design for the sequential gacha game with the reset-after-winning mechanism, we need to derive the maximum of the following function:

$$f(x) = \text{Revenue}(x, y = x) = \Pr((a)) \cdot (x + y) + \Pr((b)) \cdot x + \Pr((c)) \cdot (x + y), \quad x \in [0, 1].$$

Then we have

$$x^* = \arg\max f(x) = 0.43425853,$$
$$\max f(x) = f(x^*) = 0.5161512329820706.$$

Therefore, in the sequential gacha game with reset-after-winning mechanism, the maximum seller's revenue is 0.516.

We now investigate the sequential gacha game with a hard pity system and the succeed-after-winning mechanism. For convenience, we consider a hard pity gacha game where $N = 100, p_N = 1$ and $p_i \equiv p, \forall i < N$ and the price of the gacha pull $c = 0.01$. With the succeed-after-winning mechanism, if the buyer wins the gacha game firstly at S_N, he will be at the next state S_{N+1}.

Mathematically, $\forall i > N, p_i = p_{i \bmod N}$. To figure out the optimal design for this gacha game, we need to derive a proper probability p such that the seller can achieve the maximum revenue.

According to Lemma 3.3, the expected number of gacha pulls needed to win the above hard pity gacha game when buyer is at state S_i is

$$E(\pi_\infty, S_i) = \frac{1-(1-p)^{N-i+1}}{p}.$$

Therefore, the expected number of gacha pulls needed to win the gacha game once is

$$E_1 = E(\pi_\infty, S_1) = \frac{1-(1-p)^N}{p}.$$

According to Proposition 3, the expected number of gacha pulls needed to win the gacha game twice is

$$E_2 = H(2,1) = (1-p)^{N-1}\left(N + \frac{1-(1-p)^N}{p}\right) + \sum_{i=1}^{N-1} p(1-p)^{i-1}\left(i + \frac{1-(1-p)^{N-i}}{p}\right).$$

In the gacha game with succeed-after-winning mechanism, there are the possible scenarios:

(1) When $R_1 \geq x = E_1 \cdot c$, the buyer will first pull the gacha game until he wins. Assume that the buyer wins the gacha game for the first time at state S_i. if $R_2 \geq y = H(1, i+1) \cdot c$, the buyer will continue pulling the gacha game until he wins again. In this case, the seller can obtain a revenue of $(x+y) = (E_1 + H(1, i+1)) \cdot c$.
(2) When $R_1 \geq x = E_1 \cdot c$, the buyer will first pull the gacha game until he wins. Assume that the buyer wins the gacha game for the first time at state S_i. If $R_2 < y = H(1, i+1) \cdot c$, the buyer will quit. In this case, the seller can obtain a revenue of $x = E_1 \cdot c$.
(3) When $R_1 \geq x = E_1 \cdot c$ and $R_1 + R_2 \geq E_2 \cdot c$, the buyer will continue pulling the gacha game until he wins twice. In this case, the seller can obtain an expected revenue of $E_2 \cdot c$.
(4) Otherwise, the buyer will never pull the gacha game, and the seller can obtain zero revenue.

Here $H(t, i)$ denotes the expected number of gacha pulls needed to win the gacha game t more times when the buyer is at state S_i, which is formulated in Proposition 3. Here we have

$$H(1, i+1) = \begin{cases} \frac{1-(1-p)^N}{p}, & i = N, \\ \frac{1-(1-p)^{N-i}}{p}, & \text{otherwise.} \end{cases}$$

Similar to the previous analysis, the seller's expected revenue is

$$\begin{aligned}\text{Revenue}(p) = (1 - E_1 c) \cdot c &\left(\sum_{i=1}^{N} p^{i \bmod N}(1-p)^{i-1}(i + (1 - H(1, i+1) \cdot c) \cdot H(1, i+1))\right) \\ &+ \begin{cases} (2 + E_1 c - 2E_2 c) \cdot (E_1 c)/2 \cdot E_2 c, & E_2 \cdot c \leq 1, \\ (1 - E_2 c + E_1 c)^2/2 \cdot E_2 c, & \text{otherwise.} \end{cases}\end{aligned}$$

Using the "SLSQP" method in Python, we can obtain the maximum seller's revenue as follows:

$$p^* = \arg\max \text{Revenue}(p) = 0.01722628,$$
$$\max \text{Revenue}(p) = \text{Revenue}(p^*) = 0.5218329662856214.$$

Thus, for the sequential gacha game with the succeed-after-winning mechanism where $N = 100, p_i = 0.172, \forall i < N$ and $P_N = 1$, and the price of the gacha pull $c = 0.01$, the seller's revenue is 0.5218. ◻

Example 3. Consider that buyer is budget-constrained and get some periodical income $I = 50$ in the time frame of each banner, such as monthly salary. There are two banners in this game and the buyer's valuation of the reward in these banners are $R_1 = 100, R_2 = 50$. The price for each gacha pull is $c = 1$. Consider the banner-based gacha game where $N = 100$, $p_i = 0.01, \forall i < N$ and $p_N = 1$.

- With the reset-after-opt-out mechanism, the buyer will pull in the first banner and will never pull in the second banner. In this case, the seller's expected revenue is 39.499.
- With the succeed-after-opt-out mechanism, the buyer will first pull in the first banner. If the buyer exhausts his budget but fails to win in the first banner, the buyer's state will be inherited to the second banner, which will lower the cost to win in the second banner. Therefore, the buyer will pull in the second banner. In this case, the seller's expected revenue is 63.397.

Proof. In the first banner, the buyer has a budget of 50. According to Lemma 3.1, we have that $V_{\pi_{50}}(S_1) = 0 \geq 0$. According to Lemma 4.3, the gacha game has the whale property. Therefore, the optimal policy for the buyer in the first banner is π_{50}. Based on Lemma 3.3, we know that the expected number of gacha pulls the buyer spends on the first banner is $E(\pi_{50}, S_1) = 39.499$.

With the reset-after-opt-out mechanism, the buyer's state will be reset to S_1 in the second banner. In the second banner, according to Lemma 3.1, we have that $V_{\pi_{50}}(S_1) = -19.750 < 0$. The whale property of the gacha game implies that the optimal policy for the buyer in the second banner is π_0. Therefore, with the reset-after-opt-out mechanism, the seller's expected revenue is 39.499.

With the succeed-after-opt-out mechanism, there are two possible scenario:

(1) If the buyer wins the gacha game in the first banner within 50 gacha pulls, the buyer's state in the second banner will be S_1. Similarly, the buyer would not pull in the second banner.
(2) If the buyer does not win the gacha game in the first banner within 50 gacha pulls, the buyer's state in the second banner will be S_{51}. And in the second banner, the buyer will has another budget of 50. In the second banner, according to Lemma 3.1, we have that $V_{\pi_{100}}(S_{51}) = 10.501 > 0$. Therefore, the optimal policy for the buyer in the second banner is π_{100}, i.e., continue pulling until he wins.

Considering these two possible scenario, the expected seller's revenue in the banner-based gacha game with the succeed-after-opt-out mechanism is

$$\text{Revenue} = E(\pi_{50}, S_1) + (1-p)^{50} \cdot E(\pi_{100}, S_{51}) = 63.397.$$

Thus, when the buyer has budget constraint, the succeed-after-opt-out mechanism can help to achieve a higher seller's revenue. □

Example 4. Consider a banner-based multi-item gacha game with 2 banners. The buyer's valuations of the item in these 2 banners are $R_1 = 50$ and $R_2 = 100$. Each banner is a gacha game where $N = 100$, $p_i = 0.01, \forall i < N$ and $p_N = 1$. According to Theorem 5.2, without any subsidies, the buyer will only pull in the second banner, which will lead to the expected seller's revenue being 63.397. If the seller subsidizes the buyer as it does in the single-item gacha game, according to Theorem 6.2, the seller should give the buyer 32 free gacha pulls in the first banner and no free gacha pull in the second banner, assuming that these subsidies will encourage the buyer to pull in the first banner. However, Theorem 6.3 shows that a rational buyer will accumulate these subsidies and only buy the gacha pull in the second banner, resulting in the lower expected seller's revenue being 35.895. In this case, the subsidies lead to the buyer's grinding behavior and harm the seller's revenue.

Proof. According to Lemma 4.3, the gacha game has the whale property. In the first banner, according to Lemma 3.1, we have that $V_{\pi_\infty}(S_1) = -19.750 < 0$. Without any subsidies, the whale property of the gacha game implies that the optimal policy for the buyer in the first banner is π_0.

And in the second banner, we have that $V_{\pi_\infty}(S_1) = 36.603 > 0$. Therefore, the optimal policy for the buyer in the second banner is π_∞. The expected number of gacha pulls the buyer will spends on the second banner is $E(\pi_\infty, S_1) = 63.397$. Therefore, without any subsidies, the expected seller's revenue is 63.397.

If the seller subsidizes the buyer as it does in the single-item gacha game, according to Theorem 6.2, the seller should give the buyer $m^* = \lceil \arg\max U_s(m) \rceil = 32$ free gacha pulls in the first banner and no free gacha pull in the second banner. However, according to Theorem 6.3, since $R_1 < E(\pi_\infty) \cdot c$ and $R_2 \geq E(\pi_\infty) \cdot c$, the buyer will accumulate these subsidies and only buy the gacha pull in the second banner. According to Theorem 6.2, the expected seller's revenue is $U_s(32) = 35.895$. Therefore, in this case, the subsidies lead to the buyer's grinding behavior and harm the seller's revenue. □

F RECOMMENDATIONS ON GACHA GAME MECHANISMS

In summary, we have the following recommendations for the seller:

- A whale property gacha game is always a good choice unless the seller wants to expand the number of participants in the gacha game instead of pursuing revenue maximization.
- When selling a single item, the varying-probability gacha game with whale property and the optimal design in Theorem 4.10 is recommended. The optimal design helps the seller to achieve the maximum revenue, and the varying probability makes subsidy a useful tool for the seller to increase his revenue when the buyer's valuation is too low.
- When selling multiple items in the sequential gacha game, the reset-after-winning mechanism is recommended due to its simplicity, widespread use, and asymptotic optimality.
- When selling multiple items in the banner-based gacha game, the seller is recommended to adopt the succeed-after-opt-out mechanism, which is friendly to the buyer and can help the seller to achieve a higher revenue when the buyer is budget-constrained.

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